Proof and False Proofs

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March 2023

1. A knight always tells the truth and a knave always lies. Bob, Jerry and Tom are each either a knight or knave and know who the other knights and knaves are. Bob says everybody is a knight. Jerry says that Bob is a knave, then, shortly afterwards, says Tom is a knave. Out of these three, who is a knight and who is a knave?

Solution: Since Jerry says Bob is a knave and that Tom is a knave, either Jerry is a knight, in which case there are two knaves, or Jerry is a knave, in which case there is only one knave. However, in both cases, at least one person is a knave, so Bob must be lying when they state "everybody is a knight". Thus, Bob is a knave. This means Jerry was telling the truth, so Jerry is a knight. It follows that Tom is a knave.

2. Find the mistake in this inductive proof that $2^n > n^2$ for all $n \geq 0$.
Base case, $n = 0$: $2^0 = 1 > 0^2 = 0$.
Now we show $2^k > k^2 \implies 2^{k+1} > (k+1)^2$.

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \\ 2 \cdot 2^k &> 2k^2 \text{ by the inductive step} \\ (k+1)^2 &= k^2 + 2k + 1 \\ k^2 &> 2k + 1 \\ \implies 2k^2 &> k^2 + 2k + 1 = (k+1)^2 \\ \implies 2 \cdot 2^k &> (k+1)^2 \end{aligned}$$

so, inductively we have $2^n > n^2$ for all $n \geq 0$.
Solution: $k^2 > 2k + 1$ only holds for $k > 1 + \sqrt{2} > 2$ (when $k \geq 0$).

3. Given that $a \implies b$ and $b \implies c$ and $d \implies b$, which of the following are necessarily true?
i. $a \implies c$
ii. $a \implies d$
iii. $\neg c \implies a$
iv. $\neg c \implies \neg a$
(Hint: $p \implies q$ statements mean p being true implies q is true, and nothing more. $\neg p$ is read as the negation of p, and is the logical opposite of p, so that if p is true, $\neg p$ is false, and if p is false, $\neg p$ is true.)

Solution: i) must be true upon chaining $a \implies b$ and $b \implies c$, ii) does not always have to be true, iii) does not have to be true since it's the wrong version of the contrapositive from i) (and there is no other indication of it being true), and iv) must be true due to contraposition of the result from i).

4. What is wrong with this proof?

$$\begin{aligned}
&\int \frac{1}{x\log(x)} \\
&u = \frac{1}{\log(x)}, dv = 1/x \\
&du = \frac{-1}{x\log(x)^2}, v = \log(x) \\
\implies &\int \frac{1}{x\log(x)} = 1 + \int \frac{1}{x\log(x)} \\
\implies &0 = 1
\end{aligned}$$

Solution: The manipulation of the integrals is actually completely fine! This is because we are calculating with indefinite integrals (i.e. antiderivatives), which can hold equality *up to a constant factor*. Thus the incorrect step in this proof was assuming $0 = 1$ just because we can extract a constant factor out of $\int \frac{1}{x\log(x)}$.

5. (SMMC 2022) C1. Let A and B be two fixed positive real numbers. The function f is defined by

$$f(x, y) = min\{x, \tfrac{A}{y}, y + \tfrac{B}{x}\},$$

for all pairs (x, y) of positive real numbers. Determine the largest possible value of $f(x, y)$. (Note: once you have an answer, ensure to try and rigorously prove this solution works).
Solution: All real numbers are generated by $\frac{A}{y}$ for all real y, so we can define a new function $g(x, y) = min\{x, y, \frac{A}{y} + \frac{B}{x}\}$ whose maximum value is the same as that of $f(x, y)$.
If $x \leq y$, $g(x, y)$ is either x or $\frac{A}{y} + \frac{B}{x}$. The former is strictly increasing while the latter is strictly decreasing (over x), so the maximum value of $g(x, y)$ (when y is fixed) occurs when $x = \frac{A}{y} + \frac{B}{x}$. From here, (noting that $x \neq 0$), we get $x^2 - \frac{A}{y}x - B = 0$. Solving this quadratic equation yields $g(x, y) = x = \frac{\frac{A}{y} + \sqrt{\frac{A^2}{y^2} + 4B}}{2}$ (note the square root part is strictly greater than $\frac{A}{y}$, so we take the positive root). Note that if we now decrease y, $g(x, y)$ must increase. Thus the largest possible value $g(x, y)$ can take is when y is as low as it can be, i.e. when $x = y$.
This is similarly true when $y \leq x$ (we repeat the same argument but first fix x and then vary it to reach the same conclusion). Thus the largest value $g(x, y)$ takes is when $x = y = \frac{A}{y} + \frac{B}{x}$. So, we find $x = \frac{A}{x} + \frac{B}{x} = \frac{1}{x}(A + B) \implies x^2 = A + B$. Since x must be positive, $g(x, y)$'s maximum value is $\sqrt{A + B}$.

6. (IMO 2022 P2) Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that there exists exactly one positive real y for every positive real x such that:

$$f(x)y + f(y)x \leq 2$$

Solution: If $f(x) = \frac{1}{x}$, then $f(x)y + f(y)x = \frac{x}{y} + \frac{y}{x} \geq \frac{x}{y} \cdot \frac{y}{x} = 2$ by the AM-GM inequality. Equality is only achieved for $\frac{x}{y} = \frac{y}{x}$, or, equivalently, when $x = y$, otherwise we have a strict "greater than" inequality. Thus for $f(x) = \frac{1}{x}$, there exists only one positive real y (i.e. $x = y$) where the functional equation holds, so this is a solution.

If there exists a positive real m such that $f(m) > \frac{1}{m}$, then $y = m$ does not satisfy the functional equation since $f(m)m + f(m)m > \frac{m}{m} + \frac{m}{m} = 2$. This means there must be some $n \neq m$ such that $f(m)n + f(n)m \leq 2$. But this means that for $x = n$, $y = m$ satisfies the functional equation (note the equation's symmetric nature). Furthermore, $f(n) < \frac{1}{n}$ since if $f(n) \geq \frac{1}{n}$, then $f(n)m + f(m)n \geq \frac{m}{n} + \frac{n}{m} > 2$ (note the AM-GM inequality is strict since $m \neq n$). Thus, for $x = n$, $y = n$ is also satisfies the functional equation. Therefore we've reached a contradiction since we've found, for $x = n$, $y = m$ and $y = n$ are two distinct values which satisfy the functional equation.

If there exists a positive real m such that $f(m) < \frac{1}{m}$, then $x = m$ and $y = m$ satisfy the functional equation as $f(m)m + f(m)m < \frac{m}{m} + \frac{m}{m} = 2$. So, $f(\frac{1}{f(m)}) > f(m)$ since otherwise $x = m$ and $y = \frac{1}{f(m)}$ would also satisfy the functional equation as $\frac{f(m)}{f(m)} + f(\frac{1}{f(m)})m \leq 1 + f(m)m < 1 + \frac{m}{m} = 2$. However, this satisfies the previous scenario, so a contradiction can once again be reached. Thus $f(x) = \frac{1}{x}$ is the only solution.