

Some problemos

CPMSoc

March 2023

- 1) Is every integer greater than two the sum of two primes?
- 2) Isaiah starts with the number 0 and, every turn, randomly adds either 1 or 0. If there number gets to 5, 9 is subtracted from it. What is the expected value of the number they end up with after 9 turns?
(There is a nice solution that involves no calculations!)
- 3) (IMC 2022 First Day Problem 1) Let $f : [0, 1] \rightarrow (0, \infty)$ be an integrable function such that $f(x)f(1-x) = 1$ for all $x \in [0, 1]$. Prove that

$$\int_0^1 f(x)dx \geq 1.$$

- 4) A triangle ABC is inscribed in a circle. A line tangent to the circle at point B has a point D chosen on it such that $\angle ABE \geq \angle CBE$. Show that $\angle CBE = \angle CAB$.
- 5) Evaluate:

$$\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right).$$

- 6) Show there exists a positive Fibonacci number which is divisible by 2023
- 7) (IMO 2017 P2) Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ where, for all real x, y :

$$f(f(x)f(y)) + f(x+y) = f(xy).$$

- 8) (Simon Marais 2021 problem B2) Let n be a positive integer. There are n lamps, each with a switch that changes the lamp from on to off, or from off to on, each time it is pressed. The lamps are initially all off.

You are going to press the switches in a series of rounds. In the first round, you will press exactly 1 switch; in the second, you will press exactly 2 switches; and so on, so that in the k th round you will press exactly k switches. In each round you will press each switch at most once. Your goal is to finish a round with all of the lamps switched on.

Determine for which n you can achieve this goal.