Graph Theory (with Linear Algebra) problems

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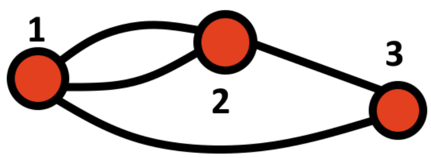
1. Evaluate the matrix product AB where $A = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 4 & 2 \\ 4 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$.

Recall that the ijth entry of the resultant matrix should be the dot product of the ith row of the first matrix with the jth column of the second matrix.

Solution: $\begin{pmatrix} 10 & 2 & 4 \\ 2 & 10 & 4 \\ 4 & 4 & 4 \end{pmatrix}$

2. How many unique length 4 paths are there between vertices 1 and 2 in the graph below? Confirm this result by raising an adjacency matrix to the 4th power. The 4th power of A can be quickly calculated by calculating A^2 and then squaring the resultant matrix, or just finding A^2 and then taking the dot product of its 1st row and 2nd column.



Solution: $\begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^4 = \begin{pmatrix} 5 & 1 & 2 \\ 1 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}^2 = \begin{pmatrix} 30 & 14 & 16 \\ 14 & 30 & 16 \\ 16 & 16 & 12 \end{pmatrix}$, so 14.

3. A directed graph is a graph where each edge connects only one way from one vertex to another, typically drawn as an arrow between them. Let the Laplacian matrix L of a simple directed graph G with n vertices be an $n \times n$ matrix where

$$L_{ij} = \left\{ \begin{array}{ll} \text{degree of vertex } v_i & \text{if } i = j \\ -1 & \text{if } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise} \end{array} \right\}.$$

Let the incidence matrix A of graph G, under some ordering of the m edges and n vertices, be an $n \times m$ matrix (n rows, m columns) such that

$$A_{ij} = \left\{ \begin{array}{ll} 1 & \text{if edge } e_j \text{ goes into vertex } v_i \\ -1 & \text{if edge } e_j \text{ goes out of vertex } v_i \\ 0 & \text{otherwise} \end{array} \right\}.$$

The transpose operation flips a matrix about the main diagonal (the one going from the top-left to the bottom-right). Formally, the transpose of M, $M^\top$, is the matrix where ${M^\top}_{ij} = M_{ji}$. This operation transforms a $n \times m$ matrix into a $m \times n$ matrix. Show that $AA^\top = L$ for all simple directed graphs G.

Solution: Denote row vector i of A as A_i. Note that $(AA^T) = A_i \cdot A_j$, we will use this to show every entry is equal to the corresponding one in L_{ij}. When $i = j$, $(AA^T)_{ij} = A_i \cdot A_i$. Since the row vector consists of only 1's and -1's, every non-zero element will be multiplied with itself and thus will add 1 to our dot product. So, $A_i \cdot A_i$ counts the total number of edges going into or out of vertex i (which is its degree).

When $i \neq j$ and there is an edge going from vertex i to vertex j, then $(AA^T)_{ij}$ will be the dot product two row vectors with -1 (edge going out of i) in the first vector being in the same position as a 1 (edge going into j) in the second, but no other position will have a non-zero element for both vectors (as G is simple, so only one edge can go from i to j). So, $A_{ij} = A_i \cdot A_j = -1$.

If none of these cases hold, then there is no edges from i to j, so no non-zero element in A_i can have the same position as a non-zero element in A_j, thus $(AA^T)_{ij} = 0$. This fully maps out L's entries as well, thus $AA^T = L$.

4. We call two graphs "isomorphic" if there's a reordering from one graph to the other that preserves their graph "structure". Informally, this occurs if we can move the vertices of one graph around and relabel them without adding, removing or moving edges and get the other graph. How many unique graphs, up to isomorphism, are there with 3 vertices and 4 edges? How about 3 vertices and n edges?

Note: formally, if graphs G and H respectively have vertex sets A, B and edge lists E, F (lists of unordered pairs of vertices, possibly with repeated entries for double edges), then they are isomorphic if there exists a bijective function $f : A \to B$ satisfying $(u, v) \in E \iff (f(u), f(v)) \in F$.

Solution: Consider the mapping between unique 3-vertex graphs, up to isomorphism, to sets (unordered) of three non-negative numbers representing the number of edges between vertices. This mapping is one-to-one is because every 3-vertex graph has three pairs, and thus three numbers corresponding to the number of edges between each pair of vertices (ordering does not matter in sets). Furthermore, every set of 3 non-negative numbers can be uniquely assigned to a 3-vertex graph (all graphs with those numbers of edges between vertices can be relabelled and thus will be the same up to isomorphism, note this does not hold generally for graphs with more than 3 vertices). The calculation of the number of combinations of non-negative numbers that add to n is left as an exercise :P (hint: consider how many ways you can choose the smallest element, then the next smallest etc.).

5. We saw an example of a connected graph that had an Euler trail (a trail where all edges are visited exactly once). It had two vertices with odd degree and the rest with even degree. Show that these conditions are sufficient to guarantee the existence of such a trail.

Solution: We start from an odd-degree vertex and go on a random trail by constantly going down an edge that hasn't been coloured and colouring it red, until there are no more uncoloured edges we can visit, at which point we will have reached the other odd-degree vertex (note that this must happen due to the parity of starting/ending/passing through vertices).

Note that the number of uncoloured edges in each vertex is even. Now, we "reverse time" and walk back, "saving" each step we take in the reverse direction until we find a vertex with at least one uncoloured edge.

We find a circuit by once again randomly choosing and colouring edges until we can't and ending back at our starting vertex (this must, again, happen due to our parity argument). Again, every vertex has an even degree. We repeat the process by "reversing time" again, walking back while "saving" the steps we took. In this way, we keep augmenting the trail with circuits we keep finding until we reach our very first starting vertex. This will exhaust all vertices in the graph because the process of saving edges to our final trail (in reverse) ensures every vertex that we've walked back from has no uncoloured outgoing edges, and since the graph is connected, our trail must therefore go through every single edge, thus being an Euler trail.

6. Let d_i be the degree of vertex i in an undirected graph. Then, the number walks of length 2 is

$$\sum_{i \in V(G)} d_i^2.$$

a) Prove this with a graph
b) Prove this with an adjacency matrix

Solution: The number of length 2 walks which go into and then out of vertex i can be found by first counting the number of edges going into i and multiplying it by the number of edges going out of it (this covers all orderings of 2 edges with i as a common vertex). In both of these cases, this is d_i edges. So, the total number of length 2 walks can be found by adding this up d_i^2 for each vertex i.

The adjacency matrix A, when calculating the sum of the entries of A^2, has, for every i, j, A_{ij} in the left matrix multiplied by every possible A_{jk}. This means a sort of convolution is present, where the sum can be reframed as calculating the sum of, for each, $A_{ij} \times d_j$. Now, each d_j is multiplied by every possible A_{kj}, which factorised once again leads to the sum of every possible d_i^2.

7. (Simon Marais 2021) Let $n \geq 2$ be an integer, and let O be the $n \times n$ matrix whose entries are all equal to 0. Two distinct entries of the matrix are chosen uniformly at random, and those two entries are changed from 0 to 1. Call the resulting matrix A.

Determine the probability that $A^2 = O$, as a function of n.

Solution: This can be reframed as a graph problem! Treat A as the adjacency matrix of a directed graph: now we wish to find the probability that a directed graph with two randomly chosen edges (that aren't the same) does not result in a length-2 walk (this is when $A^2 = O$).

The first edge has a $\frac{1}{n}$ chance of being a self-loop, resulting in a length-2 walk by looping twice on that vertex. If this does not happen, an $\frac{n-1}{n}$ chance of occurring, a length-2 walk can still be generated if the second edge is a self loop (n possibilities) or goes from the first edge's ending vertex to any of the other $n-1$ vertices, or it goes to the first edge's starting vertex from any of the other $n-1$ vertices. We may tempted to count $3n-2$ possibilities, however notice that we've double counted the edge going from the first edge's ending vertex to its starting vertex, so really there are $3n-3$ possibilities. Also note there are only n^2-1 total possibilities for where the second edge can go since the first edge already takes up a spot.

Therefore the probability that we have no length-2 walks in A^2 is

$$1-\frac{1}{n}-\frac{n-1}{n}\times\frac{3n-2}{n^2-1}=\frac{n(n+1)-(n+1)-(3n-3)}{n(n+1)}=\frac{n^2-3n+2}{n(n+1)}=\frac{(n-1)(n-2)}{n(n+1)}.$$

8. Extension: this problem is extension in the sense that it requires content not covered in the workshop, and significant use of external tools possibly unavaliable during competitions. If a graph has adjacency matrix A, and $I-A$ is invertible (I is the appropriately sized identity matrix), can we find a closed form (i.e. with no summation) for the number of length $\leq n$ paths? What other conditions must be met? In particular find a closed form for length $\leq n$ paths between vertices 1 and 4 in the adjacency matrix:

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 3 & 0 & 1 & 0 \end{pmatrix}.$$

Note that to complete this problem fully requires a lot of heavy computation, I recommend using maple if you really want to try this.
You should get $-0.1538461538-0.1401205851*0.258777175^{n+1}+0.08250910327*(-0.258777175)^{n+1}+0.1330890226*3.864328451^{n+1}+0.07836861309*(-3.864328451)^{n+1}$.

Solution: Define $\text{sum}(A)$ to be the sum of the entries of matrix A. Then we must calculate: $\text{sum}(A+A^2+\cdots+A^n)$. Due to the associative and distributive properties of matrices, we can rearrange this summation to get the same formula we would for geometric sums:

$$S=A+A^2+\cdots+A^n$$

$$S\times A=A^2+\cdots+A^n+A^{n+1}=S+A^{n+1}-A$$
$$S\times A-S=S(A-I)=A^{n+1}-A=A(A^n-I)$$
$$S=A(A^n-I)(A-I)^{-1}.$$

Thus our answer will be $\text{sum}(S)=\text{sum}(A(A^n-I)(A-I)^{-1})$.