Number theory problems

Cyril and Zac

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1 Problems

1. what is $57 \times 19 \bmod 13$.

2. Evaluate $\gcd(52, 91)$.

3. Evaluate $5^{123} \bmod 7$.

4. Prove that for all integers n with $n \geq 3$, if $2^n - 1$ is prime, then n cannot be even.

5. (Wilson's theorem) Show that $(p-1)! = -1 \pmod{p}$ for prime p.

   Hint: Consider inverses

6. Prove that among any three distinct integers we can find two, say a and b, such that the number $a^3b - ab^3$ is a multiple of 10.

7. Define the function $f(x, y)$ for positive integers x, y as:

$$f(x,y) = \left\{ \begin{array}{ll} f(y, x \bmod y) + 1 & \text{for } x, y > 1 \\ 0 & \text{else} \end{array} \right\}$$

   where $x \bmod y$ refers to the remainder after calculating $x \div y$. Find two values $x \leq y \leq 90$ for which $f(x, y)$ attains its maximum.

8. Define the sequence of integers $a_1, a_2, a_3, ...$ by $a_1 = 1$, and

$$a_{n+1} = (n + 1 - \gcd(a_n, n))a_n$$

for all integers $n \geq 1$. Prove that $\frac{a_{n+1}}{a_n} = n$ if and only if n is prime or $n = 1$. (Simon Marais 2021)