inequalities solutions

Cyril and David

July 2023

1 Introduction

All variables are positive real numbers. Prove each of the following inequalities and (if possible) find the values for which equality holds.

1. [AM-GM] $a^2b^2 + b^2c^2 + c^2a^2 \geq abc(a + b + c)$
Solution: (note that rearrangement can be directly applied :D) Take $a^2b^2 + b^2c^2 \geq 2\sqrt{a^2b^2b^2c^2} = 2ab^2c$. Performing this cyclically with the other two pairs of terms and adding them up, we obtain $2(a^2b^2 + b^2c^2 + c^2a^2) \geq 2abc(a + b + c)$

2. [Squares] $a^2 + \frac{1}{a^2} + 6 \geq 4a + \frac{4}{a}$
Solution: $a^2+\frac{1}{a^2}+2+4-4(a+\frac{1}{a}) = (a+\frac{1}{a})^2-4(a+\frac{1}{a})+4 = (a+\frac{1}{a}-2)^2 \geq 0$

3. [Triangle substitution] Let a, b, c be the sides of a triangle. Prove that $(a + b)(b + c)(c + a) \geq 8(a + b - c)(b + c - a)(c + a - b)$.
Solution: From triangle substitution we may write $a = y+z, b = z+x, c = x + y$ for positive real numbers x, y, z. Then,

$$\begin{aligned}(y + z + z + x)(z + x + x + y)(x + y + y + z) &= (x + y + 2z)(2x + y + z)(x + 2y + z)\\ &\geq 4 \cdot \sqrt[4]{xyz^2} \cdot 4 \cdot \sqrt[4]{x^2yz} \cdot 4 \cdot \sqrt[4]{xy^2z}\\ &= 64\sqrt[4]{x^4y^4z^4}\\ &= 64xyz\\ &= 8(2z)(2x)(2y)\\ &= 8(a + b - c)(b + c - a)(c + a - b)\end{aligned}$$

Note: the motivation for using AM-GM is because we want a "product" of things (namely, $64xyz$), and we use four variables since this "strongly" retains a consistent equality condition (if we instead picked $x, y, 2z$, we'd have an equality condition of $x = y = 2z$, however this isn't consistent with the other equality conditions of $x = 2y = z$ and $2x = y = z$, thus "losing" some information).

4. [Rearrangement inequality] $a^3 + b^3 + c^3 \geq a^2b + b^2c + c^2a$
Solution: WLOG $a \geq b \geq c$, then $a^2 \geq b^2 \geq c^2$. The LHS is the

maximal pairing of these sequences, and the RHS is some permutation, so the rearrangement inequality directly follows.

5. [Cauchy Schwartz]

$$(a+b+c)\left(\frac{1}{b+c}+\frac{1}{a+c}+\frac{1}{a+b}\right)\geq\frac{9}{2}$$

Solution: To use Cauchy schwartz, we aim to have a product between two sum of squares on the LHS and a dot product on the RHS. Here we have a setup where the terms can be easily cancelled out (if we first duplicate the LHS terms):

$$\begin{aligned}LHS &= \frac{1}{2}(b+c+a+c+a+b)\left(\frac{1}{b+c}+\frac{1}{a+c}+\frac{1}{a+b}\right)\\ &\geq \frac{1}{2}\left(\frac{\sqrt{b+c}}{\sqrt{b+c}}+\frac{\sqrt{a+c}}{\sqrt{a+c}}+\frac{\sqrt{a+b}}{\sqrt{a+b}}\right)^2\\ &= \frac{9}{2} = RHS\end{aligned}$$

6. [Homogenisation] $a^2+b^2+c^2 \geq a+b+c$ when $abc=1$
Solution Degree of LHS is 2 and the degree of RHS is 1. Since the degree of abc is 3, we try multiplying the RHS by $\sqrt[3]{abc}$. The form of the new RHS indicates AM-GM with three variables. trying a^2, ab, ac, we find $a^2+ab+ac \geq 3\sqrt[3]{a^4bc} = 3a\sqrt[3]{abc}$. Performing this cyclically with b and c gives us $a^2+b^2+c^2+2(ab+bc+ca) \geq 3\sqrt[3]{abc}(a+b+c)$. We then obtain our desired result by noticing $a^2+b^2+c^2 \geq ab+bc+ca$.

7. [Jensen's inequality]

$$\frac{1}{1+ab}+\frac{1}{1+bc}+\frac{1}{1+ca}\leq\frac{3}{4} \text{ when } abc=a+b+c$$

Hint: $\frac{x}{s+x}$ is concave for positive s
Solution: The ab, bc and ca in the denominators may motivate us to convert them to $\frac{a+b+c}{c}, \frac{a+b+c}{a}, \frac{a+b+c}{b}$ respectively (recall $a+b+c=abc$), and setting $s=a+b+c$ gives us $LHS = \frac{a}{s+a}+\frac{b}{s+b}+\frac{c}{s+c}$. Since $\frac{x}{s+x}$ is concave, we use Jensen's inequality to write:

$$3\left(\frac{1}{3}\left(\frac{a}{s+a}+\frac{b}{s+b}+\frac{c}{s+c}\right)\right)\leq 3\left(\frac{\frac{s}{3}}{s+\frac{s}{3}}\right)=3\left(\frac{\frac{1}{3}}{\frac{4}{3}}\right)=\frac{3}{4}$$

8. a, b, c are positive integers such that $a^2b^3c^4=1$. Find the minimum value of $a+b+c$ (you may use indices in your answer).
Solution: Using AM-GM splitting to write $a+b+c = 2\frac{a}{2}+3\frac{b}{3}+4\frac{c}{4}$, we find $a+b+c \geq \sqrt[9]{\frac{a^2b^3c^4}{2^23^34^4}} = \frac{1}{\sqrt[9]{2^{10}\times 3^3}}$. Equality is obtained when $\frac{a}{2}=\frac{b}{3}=\frac{c}{4}$ which you may verify to obtain the minimum value.

9. $(a+b)^4 \leq (5a^2+b^2)(a^2+2b^2)$
Solution: Using Cauchy Schwartz:

$$(a+b)^4 = (a^2+2ab+b^2)^2 = (a \cdot a+2a \cdot b+b \cdot b)^2 \leq (a^2+4a^2+b^2)(a^2+b^2+b^2) = (5a^2+b^2)(a^2+2b^2)$$

10. $$\frac{a^8+b^8+c^8}{a^3b^3c^3} \geq \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

Solution: Note that $LHS = \frac{a^8}{a^3b^3c^3}+\frac{b^8}{a^3b^3c^3}+\frac{c^8}{a^3b^3c^3}$. WLOG $a \geq b \geq c$, then $\frac{1}{c^3} \geq \frac{1}{a^3}$. a^8, c^8 are maximally paired with $\frac{1}{c^3}, \frac{1}{a^3}$, so swapping gives us

$$LHS \geq \frac{a^8}{a^6b^3}+\frac{b^8}{a^3b^3c^3}+\frac{c^8}{b^3c^6}$$

Doing this two more times swapping it for b and c then a and b gives us

$$LHS \geq \frac{a^8}{a^9}+\frac{b^8}{b^9}+\frac{c^8}{c^9} = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

11. $(a+b)(b+c)(c+a) \geq 8abc$
Solution: See if you can find it in our workshop slides :P

12. $ab+bc+cd+da \geq a^b b^c c^d d^a$ when $a+b+c+d=1$
Solution: As noted at the bottom of the Jensen's inequality slide for this workshop, the inequality also holds for any expectation (i.e. a weighted average). If we treat event a as having probability b, b with probability c, c with d and d with a, note that the probabilities add to $a+b+c+d=1$, so this is a valid probability function. Then, since ln is concave,

$$\ln(ab+bc+cd+da) \geq b\ln(a)+c\ln(b)+d\ln(c)+a\ln(d) = \ln(a^b b^c c^d d^a)$$

Applying the exponential to both sides gives us our desired inequality.