inequalities problems

Cyril and David

July 2023

1 Introduction

All variables are positive real numbers. Prove each of the following inequalities and (if possible) find the values for which equality holds.

1. [AM-GM] $a^2b^2 + b^2c^2 + c^2a^2 \geq abc(a + b + c)$
2. [Squares] $a^2 + \frac{1}{a^2} + 6 \geq 4a + \frac{4}{a}$
3. [Triangle substitution] Let a, b, c be the sides of a triangle. Prove that $(a + b)(b + c)(c + a) \geq 8(a + b - c)(b + c - a)(c + a - b)$.
4. [Rearrangement inequality] $a^3 + b^3 + c^3 \geq a^2b + b^2c + c^2a$
5. [Cauchy Schwartz]
$$(a + b + c)\left(\frac{1}{b + c} + \frac{1}{a + c} + \frac{1}{a + b}\right) \geq \frac{9}{2}$$
6. [Homogenisation] $a^2 + b^2 + c^2 \geq a + b + c$ when $abc = 1$
7. [Jensen's inequality]
$$\frac{1}{1 + ab} + \frac{1}{1 + bc} + \frac{1}{1 + ca} \leq \frac{3}{4} \text{ when } abc = a + b + c$$
Hint: $\frac{x}{s+x}$ is concave for positive s
8. a, b, c are positive integers such that $a^2b^3c^4 = 1$. Find the minimum value of $a + b + c$ (you may use indices in your answer).
9. $(a + b)^4 \leq (5a^2 + b^2)(a^2 + 2b^2)$
10. $$\frac{a^8 + b^8 + c^8}{a^3b^3c^3} \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$
11. $(a + b)(b + c)(c + a) \geq 8abc$
12. $ab + bc + cd + da \geq a^b b^c c^d d^a$ when $a + b + c + d = 1$