generating functions problems

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1. Find
$$\sum_{n=1}^{\infty} \frac{n}{2^n}$$

   Solution:
$$\begin{aligned}
\sum_{n=1}^{\infty} nx^{-n} &= -x\sum_{n=1}^{\infty} -nx^{-n-1} \\
&= -x\sum_{n=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x}x^{-n} \\
&= -x\frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n=1}^{\infty} x^{-n}\right) \\
&= -x\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{1-\frac{1}{x}}\right) \\
&= \frac{x}{(x-1)^2}
\end{aligned}$$

   Therefore $\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{2}{1^2} = 4$

2. How many ways can 30 indistinguishable votes be distributed across 4 different candidates?
   Solution: Each candidate can have $0, 1, 2, \dots$ votes, so we can express the answer to this as the coefficient of the $x^3$0 term in the expansion of $(1 + x + x^2 + \cdots)^4 = (1-x)^{-4}$. Using the generalised binomial theorem, we find our answer is $\binom{-4}{3}0 = \binom{33}{30}$.

3. Find a closed form expression for the nth Fibonacci number using generating functions (hint: have a look at our generating functions workshop slides ;))
   Solution: We found that the generating function for the Fibonacci numbers is $\frac{x}{1`-x-x^2}$, which we can separate using partial fractions to write $\frac{1}{\sqrt{5}}\left(\frac{1}{1-\frac{1}{\phi}x} + \frac{1}{1-\phi x}\right)$, where $\phi = \frac{1+\sqrt{5}}{2}$ (the golden ratio). To obtain the

x^n coefficient, we simply note the two parts inside the parantheses each generate a geometric series, so $F_n = \frac{1}{\sqrt{5}}(\phi^n - \frac{1}{\phi^n})$.

4. Let p_n be the nth odd prime number. Show that

$$\prod_{n=1}^{\infty}\left(\frac{p_n^2}{p_n^2-1}\right) = \frac{\pi^2}{8}$$

Solution: Note that $\frac{p^2}{p^2-1} = \frac{1}{1-\frac{1}{p^2}} = 1 + \frac{1}{p^2} + \frac{1}{p^4} + \cdots$. Applying this to the product, we see we have a product of multiple geometric series each with the ratio of a reciprocal of a different odd prime's square. We can treat this as a factorisation, where from each factor we can choose 1, or $\frac{1}{p^2}$, or $\frac{1}{p^4}, \ldots$. Thus the original equation generates a series of the sum of squares of reciprocals of odd integers (i.e. $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$). Then:

$$\begin{aligned}\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots - \frac{1}{2^2} - \frac{1}{4^2} - \cdots \\ &= \frac{\pi^2}{6} - \frac{\pi^2}{24} = \frac{4\pi^2 - \pi^2}{24} \\ &= \frac{\pi^2}{8}\end{aligned}$$

5. The Catalan numbers are generated by the generating function $C(z) = \frac{1-\sqrt{1-4z}}{2z}$. Find a closed formula for the nth Catalan number.
Solution: Note we can use the generalised binomial theorem for fractional powers! So now we just have to find the coefficient of z^{n+1} and divide by 2, which gives:

$$\begin{aligned}-\frac{1}{2z}\binom{\frac{1}{2}}{n+1}1^{\frac{1}{2}-n-1}(-4z)^{n+1} &= -\frac{1}{2}\frac{1\cdot -1\cdot -3\cdots(-2n+1)}{(n+1)!\cdot 2\cdot 2^n}(-4)^{n+1}z^n \\ &= \frac{1\cdot 3\cdots(2n-1)}{2^n(n+1)!}\cdot 4^n z^n \cdot \frac{2\cdot 4\cdots(2n)}{2^n n!} \\ &= \frac{(2n)!}{4^n n!(n+1)!}\cdot 4^n z^n = \frac{1}{n+1}\frac{(2n)!}{n!n!} \\ &= \frac{1}{n+1}\binom{2n}{n}\end{aligned}$$

6. How many ways can a $3\times n$ grid be tiled with 2×1 dominoes?
Solution: We write a two-variable recurrence relation and solve for it using generating functions (see `https://en.wikipedia.org/wiki/Generating_function#Applications`)

7. [IMO Shortlist 1998] Suppose $a_0, a_1, a_2, \cdots$ is an increasing sequence of non-negative integers such that every non-negative integer can be written as $a_i + 2a_j + 4a_k$, for a unique triple (a_i, a_j, a_k). Find a_{1998}.

Solution: Denote the generating function for the z^{a_n} terms as $G(z)$. Then $G(z)G(z^2)G(z^4) = \frac{1}{1-z}$. Note that $G(z^2)G(z^4)G(z^8) = \frac{1}{1-z^2}$, so $\frac{G(z^4)G(z^2)G(z)}{G(z^8)G(z^4)G(z^2)} = \frac{G(z)}{G(z^8)} = \frac{1-z^2}{1-z} = 1+z$. We may repeat this when substituting $z \to z^8$ to obtain $\frac{G(z^8)}{G(z^64)}$, and so on. Eventually, we have an 8^k which is larger than any coefficient which we may want to find, thus we realise that $G(z) = (1+z)(1+z^8)(1+z^{64})\cdots$. The $1998th$ lowest-order z^k term is the $1998th$ smallest number which contains only $1's$ and $0's$ in its base 8 representation, so our answer is the sum of the cubes of the distinct powers of two whose sum is 1998.