functional equations

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1 Problems

Find a function $f : \mathbb{R}^+ \to \mathbb{R}$ where $f(xy) = \frac{f(x)f(y)}{f(x)+f(y)}$
Find all functions which satisfy each property:

1. $f : \mathbb{R} \to \mathbb{R}, f(x+3) = x^2 - 3x$
   Solution: Substitute $x = x - 3$:
$$f(x) = (x-3)^2 - 3(x-3) = (x-3)(x-6)$$

2. $f : \mathbb{R}^+ \to \mathbb{R}^+, f(x) + 2f\left(\frac{1}{x}\right) = x$
   Solution: Substitute $x = \frac{1}{x}$:
$$f\left(\frac{1}{x}\right) + 2f(x) = \frac{1}{x}$$
   Multiplying this by two and subtracting the previous equation:
$$3f(x) = \frac{2}{x} - x \implies f(x) = \frac{2}{3x} - \frac{x}{3}$$

3. $f : \mathbb{R} \setminus \{-1, 1\} \to \mathbb{R}, f(x)^2 f(\frac{1-x}{1+x}) = x$

Solution: Let $g(x) = \frac{1-x}{1+x}$, and note that since $g(g(x)) = \frac{1-\frac{1-x}{1+x}}{1+\frac{1-x}{1+x}} = \frac{1+x-(1-x)}{1-x+1+x} = \frac{2x}{2} = x$, so g is cyclic with order 2. Thus we may write two equations after substituting $x = \frac{1-x}{1+x}$:

$$f(\frac{1-x}{1+x})f(x)^2 = x$$

$$f(\frac{1-x}{1+x})^2 f(x) = \frac{1-x}{1+x}$$

Dividing the square of the first by the second gives us:

$$f(x)^3 = \frac{x^2(1+x)}{1-x} \implies f(x) = \sqrt[3]{\frac{x^2(1+x)}{1-x}}$$

4. $f : \mathbb{Z} \to \mathbb{R}, f(x+y) = f(x) + 2xy + f(y), f$ is continuous

Solution: The terms appear to very closely resemble the expansion of $(x+y)^2$, so we know that $f(x) = x^2$ is a solution. To find all other solutions, we try $f(x) = g(x) + x^2$, and substitute back in to find: $g(x+y) + x^2 + 2xy + y^2 = g(x) + x^2 + 2xy + g(y) + y^2$, so we obtain $g(x+y) = g(x) + g(y)$. This is the functional equation seen in the slides before (known as Cauchy's functional equation), which we know, since f, and thus g, is continuous, has $g(x) = Ax$ as its only solution for all real A. So we find the complete solution set to $f(x)$ is $f(x) = x^2 + Ax$ for all real A.

5. $f : \mathbb{R} \to \mathbb{R}, f(x)y + f(x)f(y) = f(2f(x)f(y))$

Solution: Exploiting the symmetry of the equation by noticing $f(x)y$ is the only asymmetrical equation allows us to write two equations after substituting $x = y$:

$$f(x)y + f(x)f(y) = f(2f(x)f(y))$$

$$f(y)x + f(x)f(y) = f(2f(x)f(y))$$

Subtracting these equations, cancelling terms and rearranging shows that $f(x)y = f(y)x$. Either $f(x) = 0$ for all real nonzero x (note we may also obtain $f(0) = 0$ by substituting $x = 0$), or we may substitute $y = c$ such that c and $f(c)$ is nonzero. Then, $f(x) = \frac{f(c)}{c}x = Ax$ for some real x. Plugging this back into our functional equation: $Axy + A^2xy = 2A^3xy \implies xy(2A^3 - A^2 - A) = xy(A(A-1)(2A+1))$, so we find $0, x, -\frac{x}{2}$ all satisfy the functional equation.

6. $f : \mathbb{R} \to \mathbb{R}, f(x^2 + y) = f(x^{27} + 2y) + f(x^4)$
Solution: Though this looks ugly, remember that substitutions are made to cancel terms: notice the LHS and RHS left terms are the only ones with a y term, thus we may be motivated to try and cancel them by finding when $x^{27} + 2y = x^2 + y$. Solving, we find $y = x^2 - x^{27} = x^2(1 - x^{25})$. Substituting $y = x^2 - x^{27}$, we cancel out these two terms and are left with $f(x^4) = 0$. This only gives us $f(x) = 0$ for non-negative x, however notice we may now write $f(x^{27} + 2y) = f(x^2 + y)$. Substituting $y = 0$, we find $f(x^{27}) = f(x^2) = 0$, and since x^{27} takes on all real values, we conclude $f(x) = 0$ is the only solution.

7. $f : \mathbb{R} \to \mathbb{R}, f$ is continuous, and $f(x^2) = xf(x)$
Solution: For non-negative x, $f(x) = \sqrt{x}f(\sqrt{x}) = \sqrt{x}\sqrt[4]{x}f(\sqrt[4]{x}) = f(x^{\frac{1}{2^n}})x^{\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n}})$. Taking the limit as $n \to \infty$, we notice $\frac{1}{2} + \frac{1}{4} + \cdots = 1$, and that $\frac{1}{2^n} \to 0$. Since f is continuous, it must hold that $f(x) = f(1)x$. Also, note that for $x < 0$, $f(x) = \frac{f(x^2)}{x} = \frac{f(1)x^2}{x} = f(1)x$. So, we find (with verification) that $f(x) = Ax$ for all real A is a complete solution set.

8. $f : \mathbb{R}^+ \to \mathbb{R}^+, f(f(x)) = 6x - f(x)$
Solution: We have a higher order term $(f(f(x)))$ expressed in

terms of lower order ones (x and $f(x)$), so we are motivated to try $x = f(x)$: $f(f(f(x))) = 6f(x) - f(f(x)) = 6f(x) - 6x + f(x) = 7f(x) - 6x$. This allows us to quickly realise that repeated substitution can be expressed in terms of $f(x)$ and x. First, define $f^n(x)$ as n compositions of f evaluated at x. Then, we write $f^n(x) = a_n f(x) + b_n x$, $f^2(x) = -f(x) + 6(x)$ and $f^{n+1}(x) = f^n(f(x)) = a_n f(f(x)) + b_n f(x) = 6a_n x + (b_n - a_n)f(x)$. We may then write a recurrence relation:

$$a_0 = x, a_n = f^n(x)$$

$$a_{n+2} = f(f(f^n(x))) = 6a_n - f(a_n) = 6a_n - a_{n+1}$$

This gives us the characteristic equation $x^2+x-6 = (x+3)(x-2)$, thus allowing us to write $a_n = A2^n + B(-3)^n$ for some functional terms (independent of n) A, B. However, note that since $(-3)^n$ grows faster than 2^n, if B is a non-zero function then there exists a value of $x \in \mathbb{R}^+$ where we can choose n so that $a_n < 0$, which is a contradiction. Thus, $B = 0$. Furthermore, $x = a_0 = A$. Therefore, $a_1 = f(x) = 2^1 A = 2x$ (which we may verify satisfies the functional equation).

9. $f : \mathbb{R} \to \mathbb{R}, f(f(x)^2 + f(y)) = xf(x) + y$
Solution: Substituting $x = 0$ gives us $f(f(y)) = y$ for all real y, from which we can deduce injectivity (if $f(a) = f(b)$, substitute $y = a$ and $y = b$ and equate the equations to find $a = b$). Substituting $x = f(x)$ gives us $f(f(f(x))^2 + f(y)) = f(x)f(f(x)) + y$ which, after using the previous result that f is an involution (or cyclic with order 2), gives us $f(x^2 + f(y)) = xf(x) + y$. The RHS is equivalent to the RHS of our initial functional equation, so equating and removing the enclosing f's (legal as we've proved injectivity) gives us:

$$f(x^2 + f(y)) = f(f(x)^2 + f(y))$$

$$x^2 + f(y) = f(x)^2 + f(y) \implies f(x) = \pm x$$

To show that the function is always one of the signs, suppose $f(a) = a$ and $f(b) = -b$, and substitute $x = a, y = b$ to find $f(a^2 - b) = a^2 + b$. Either $f(a^2 - b) = -a^2 + b = a^2 + b \implies a = 0$, or $f(a^2 - b) = a^2 - b = a^2 + b \implies b = 0$: in both cases, one of the values must be 0 and thus not actually have a different sign. Thus the only solutions to the functional equation are $f(x) = x$ and $f(x) = -x$ (substitute to verify)