Competitive Programming and Mathematics Society

Mathematics Workshop

Proof and False Proofs

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Welcome

- Next mathematics workshops in week 10?
- Slides will be uploaded on website (unswcpmsoc.com)

Attendance form

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Proof

- A proof is a set of logical statements that starts at assumptions and arrives at conclusions
- While a computer program might produce something, or a mathematical object, like a matrix, might represent some particular idea, a proof shows something to be fundamentally true.

Proof

- Any proof must assume things. If we assume things that eat grass fly, and that cows eat grass, then we may prove cows fly. This of course is not actually true, but if the axioms were true, so are the conclusions.
- After learning logic, we would expect to get absolutely everything right, right?

"""""Proof"""""

- Proving that 1 is the largest number.
- Suppose the opposite, that 1 is not the largest. The largest is $n > 1$. Then $n^2 < n$, as n is the largest.
- Then $n^2 - n < 0 \implies n(n-1) < 0$, but $n > 0$, and $n > 1$, so $n(n-1) > 0$, a contradiction.
- Therefore, 1 is the largest number...

Why do we get questions wrong?

- There are two ways to lose marks in maths. Not finding a solution, or presenting an incorrect solution.
- If you make an incorrect solution, you may not continue working on the problem, even if you made a simple mistake, which can be devastating, so we want to know how to look over our working to make sure we are confident in our answers.

Why do we get questions wrong?

- We will cover a number of things to look over when making or checking proofs:
  - False assumptions.
  - Poor logic, converses, inverses, and contrapositives.
  - Manipulation of equations.
  - Edge cases and induction.
  - Subtle intuitive errors in advanced maths: calculus and beyond.

False Assumptions

- Lets investigate the example from before.
- Suppose 1 is not the largest number. The largest is $n > 1$. Then $n^2 < n$, as n is the largest.
- Then $n^2 - n < 0 \implies n(n-1) < 0$, but $n > 0$, and $n > 1$, so $n(n-1) > 0$, a contradiction.
- Therefore, 1 is the largest number...
- What is wrong here?

False Assumptions

- Lets investigate the example from before.
- Suppose 1 is not the largest number. The largest is $n > 1$. Then $n^2 < n$, as n is the largest.
- Then $n^2 - n < 0 \implies n(n-1) < 0$, but $n > 0$, and $n > 1 \implies (n-1) > 0$, so $n(n-1) > 0$, a contradiction.
- Therefore, 1 is the largest number...
- What is wrong here?
- We assumed that because 1 is not the largest number, there exists a largest number $n > 1$, when in fact there is no reason to believe there exists a "largest" number.
- When doing proofs by contradiction, we must only assume the negation, and nothing more. We must only assert that there exists a number greater than 1, not that there is a "largest number"
- Pretty much all mistakes can be re-described as a form of assuming something falsely, literally saying it is true when it is not.

Poor Logic

- Consider the following (obviously wrong) proof that 1=2:
  - Begin with our hypothesis that 1=2
  - Multiply both sides by 0: 0=0
  - This statement is true
  - Therefore, our original statement is true.
- What exactly *is* wrong with our proof though?
- Typically multiplying by a common factor is routine when simplifying an equation in algebra, so why doesn't it work here?
- This is what we will spend the next few slides building towards

Implication

- When making a proof, we write it as a series of statements, each of which might be true or false. What relationship connects one statement to the next?
- Statements are related by being what is known as a sufficient-necessary pair.
- Consider the following two statements:
  1. I live in Sydney
  2. I live in Australia
- Statement 1 is a sufficient condition for statement 2. Knowing 1 is true gives us enough information to know that 2 is true.
- Statement 2 is a necessary condition for statement 1. 1 cannot be true without 2 being true.
- Note the inherent asymmetry between statements A and B. This asymmetry exists throughout many proofs and is why reversing a proof typically isn't valid.

What it means to prove something

- Most proofs come down to establishing a relationship of the form $A \implies B$, where A is the sufficient condition and B is the necessary condition.
- If A is something that is obviously true, then we know that B must be true too. (This is known as direct proof)
- If B is something that is obviously false, then we know that A must be false too. (This is known as proof by contradiction)
- It doesn't work the other way around. Knowing you live in Australia does not prove that you live in Sydney.

Working backwards

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- A proof follows a chain of implications $A \implies B \implies C \implies D \implies E$, where E is what you want to prove.
- However, when trying to prove a known statement, you know what E is, but don't know what a good choice of A might be.
- It is appealing to use a chain like $E \impliedby D \impliedby C \impliedby B \impliedby A$ where you start with what you want to prove and get to a known fact.
- However you must make sure that when reasoning backwards, the implications flow in reverse, in other words the necessary conditions are before the sufficient conditions.
- So how can you determine whether the implications are flowing forwards or backwards?

Equations

- When manipulating equations, a common strategy is to "do something to both sides of the equation".
- This can be written as going from $x = y$ to $f(x) = f(y)$ for some function f.
- This is a forward implication. The former equation is sufficient for the latter equation to be true.
- But when reverse reasoning, we want backwards implications. So the above manipulation is not always valid.
- So when does applying the same function to both sides, how do we know when it's a backwards implication too?
- i.e. what functions f have the property $x = y \impliedby f(x) = f(y)$?

Injective Functions

- An injective function (also known as an injection) is a function that never outputs the same number for multiple inputs.
- Even non-injective functions may be injective over a more limited domain.
  - e.g. $f(x) = x^2$ is not injective, yet if we were to only restrict x to the negatives, then it is injective.
- The important fact that if it is known that x falls within a domain for which we know that f is injective, then $x = y \impliedby f(x) = f(y)$.
- Be very careful that you only stick to injective functions when backward reasoning!

So what was wrong with our "proof"?

- Let's go back to the proof that 1=2:
  - Begin with our hypothesis that 1=2
  - Multiply both sides by 0: 0=0
  - This statement is true
  - Therefore, our original statement is true.
- We can see this is a "backwards reasoning" type proof, as we start with the conclusion and work backwards to get a true statement, 0=0.
- However this means we are only limited to using injective functions when deriving equations
- The main function we used here was $f(x) = 0 \times x$
- But this function is not injective!
- So that means our proof was invalid.

Common injections and non-injections

- Addition or subtraction by any number is injective
- Multiplication by any non-zero number is injective
- Multiplication by 0 is NOT injective
- For real numbers:
  - Odd powers (x^3, x^5, etc) are injective
  - Even powers (x^2, x^4, etc) are not injective, but are injective over only the positives, or only the negatives
  - Exponentials and logarithms are injective

Edge cases and induction - Horses

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- Suppose any set of n horses must be the same colour.
- Take a set of n horses, which of course must be the same colour, as assumed.
- Removing one horse, we see a set of $n-1$ horses with all the same colour.
- Add a new horse to this $n-1$ size set to get a set of n horses.
- Now, all these horses must be of the same colour, because "any set of n horses must be the same colour". Thus the horse added is of the same colour as the rest of the horses.
- If we add back the horse from before we see that any set of $n+1$ horses must be of the same colour.
- If $n=1$, all horses in a set of size 1 must be of the same colour.
- Therefore, extrapolating, we get that all horses in any set of horses must be the same colour.

Edge cases and induction - Horses

- This clearly breaks at $n = 2$, but what was wrong with the proof? Why can we not extrapolate from 1 to 2 horses?
- Let us run through with $n = 1$:
- We have 1 horse, which is the same colour as itself.
- Removing one horse, we see a set of 0 horses with all the same colour.
- Add a new horse to get a set of 1 horses.
- Now, all these horses must be of the same colour. Thus the horse added is of the same colour as the rest of the horses.
- Here the proof breaks down. In a set of 0 horses, all horses in the set are whatever colour we want, they could be purple for all we care.
- If we add back the horse from before, our proof breaks entirely - we see that the horses need not be the same colour, just because they were the same colour as no other horses.
- The natural wording of this proof masked how this proof secretly assumes $n > 1$, so that the inductive step (extrapolating from $n = 1$) never works.

Edge cases and induction

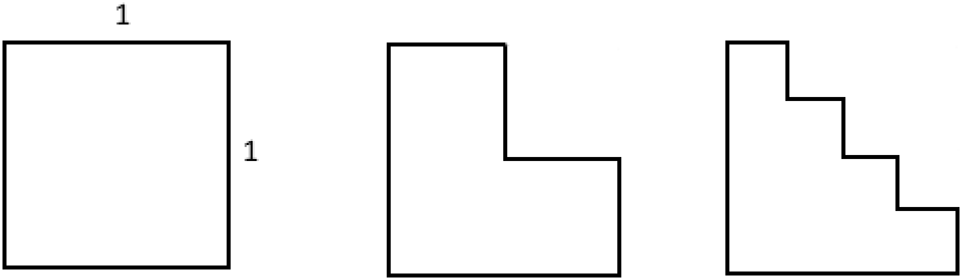
- Always check edge cases (extreme values) which may behave differently to how you expect.
- An inductive proof requires both the base case (initial proof), and the inductive step, which must apply at the base case and beyond.

Extras: calculus and beyond

- Why do we make new mistakes as maths gets more advanced if we know about all these logical things?
- Because we are fallible when lots of moving parts are involved, like in calculus, which we first may have learnt in high school with very little rigor.
- To understand math, intuition is often sufficient. To do math very well, we need to know math in a lot of depth.

Extras: calculus and beyond

- Consider the shapes below each with perimeter 4.



- The as we add more and more "steps" here, this shape approaches a right triangle with perimeter $2 + \sqrt{2} = 4$, so $2 = \sqrt{2}$.
- Why is this false? Hint: think hard about what we mean by perimeter and length here.

Attendance form :D

Further events

Please join us for:

- Maths workshop in two weeks
- Social session tomorrow
- Programming workshop next week?