Competitive Programming and Mathematics Society

Mathematics Workshop

Functional Equations

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Welcome

- Programming workshop next week
- Mathematics workshop the week after
- Try some problems on the sheet or https://t2maths.unswcpmsoc.com/
- Slides will be uploaded on website (unswcpmsoc.com)

Attendance form

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Functional Equations

- A functional equation is an equation where rather than searching for an unknown number or value, we look for an unknown function.
- For example, we may wish to find some or all functions satisfying certain properties like:
- $f(xy) = f(x)f(y)$
- $f(xy) = f(x+y)$
- $f(x) + f(y) = f(xy)$
- Solving functional equations can get very bashy, but each step can be reasonable.

Functional Equations Example

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- Find all functions $f : \mathbb{Q} \to \mathbb{Q}, f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{Q}$.

Functional Equations Solution

- Find all functions $f : \mathbb{Q} \to \mathbb{Q}, f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{Q}$.
- From other areas of math we may recognise this relationship is linear, and hence guess the set of functions $f(x) = cx$ for some real c. How can we show this covers all solutions?

Functional Equations Solution

- Find all functions $f : \mathbb{Q} \to \mathbb{Q}, f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{Q}$.
- This is a famous and important functional equation known as Cauchy's functional equation.
- From other areas of math we may recognise this relationship is linear, and hence guess the set of functions $f(x) = cx$ for some real c. How can we show this covers all solutions?
- We can consider an arbitrary input $a/b = (1+1+1+...)/b$.
- $f(a/b) = f(1/b) + f((a-1)/b) = f(1/b) + f(1/b) + f((a-2)/b) = af(1/b)$.
- We also know $f(1/b) + f(1/b) + ... + f(1/b)$ b times gives $f(1)$.
- We then get $f(a/b) = (a/b)f(1)$. Since $f(1)$ is constant and arbitrary, we get the desired solution.
- Note we cannot break a into 1's if it is not an integer.

Function Properties Refresher

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- We notice our proof in the last problem relied on use of rationals as specified in the question description. Function definitions are often denoted in the form $f : A \to B$, along with other properties. Here A and B are sets named the domain and codomain respectively. The function f has a value in B associated with every value in A, but not necessarily every value in B is associated with a value in A.
- Changing the domain and codomain can significantly alter a functional equation, from being trivial to nearly impossible, so pay attention the these specifications!

Substitutions

- A common idea in functional equations is substitution of values. For particular values this may include substituting 0 or 1, as these often give nice additive or multiplicative properties, or where multiple variables are involved, say x and y, we may make them equal, or make $x = -y$, or $x = 1/y$, or any number of other useful substitutions depending on the function properties.

Substitutions Example 1

- Find all continuous functions $f : \mathbb{R} \to \mathbb{R}, f(f(x) + y) = f(x^2 - y) + 4f(x)y$
- What substitutions might help? List them!

Substitutions Example 1 Solution

- Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$, $f(f(x) + y) = f(x^2 - y) + 4f(x)y$
- What substitutions might help? List them!
- A particularly nice substitution will cancel things out in the equation... Examples here could be $x^2 = y$ or $y = -f(x)$. In this case both substitutions prove helpful.
- We get $f(f(x) + x^2) = f(0) + 4f(x)x^2$ and $f(0) = f(x^2 + f(x)) - 4f(x)^2$.

Substitutions Example 1 Solution

- Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$, $f(f(x) + y) = f(x^2 - y) + 4f(x)y$
- We get $f(f(x) + x^2) = f(0) + 4f(x)x^2$ and $f(0) = f(x^2 + f(x)) - 4f(x)^2$.
- We now have two equations with an annoying $f(x^2 + f(x))$ term, so we may solve simultaneously to remove this, yielding:
- $f(0) = f(0) + 4f(x)x^2 - 4f(x)^2$.
- This results in $f(x)^2 = f(x)x^2$, so that either $f(x) = 0$ or $f(x) = x^2$, and we are done.
- Note here that the specified continuity condition prevents annoying behaviour for piece-wise functions which are sometimes 0 and sometimes x^2, which would be far more annoying to deal with.

Substitutions Example 2

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- Find all functions $f : \mathbb{R} \to \mathbb{R}$, $f(x + y^2) = f(y) + f(x^2 + y)$.

Substitutions Example 2 Solution

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- Find all functions $f : \mathbb{R} \to \mathbb{R}, f(x + y^2) = f(y) + f(x^2 + y)$.
- This one isn't too bad, we can respectively substitute $x = 0$ then $y = 0$ and get:
- $f(y^2) = f(y) + f(y) = 2f(y)$.
- $f(x) = f(0) + f(x^2)$.
- Relabelling x as y in the second equation because x is simply a dummy variable we have:
- $f(y) = f(0) + f(y^2)$.
- Removing that irritating $f(y^2)$ term simultaneously we find $f(y) = -f(0)$. We want to now check if there is a restriction on possible values of $f(0)$.
- We sub $x = y = 0$ and get $f(0) = f(0) + f(0) \implies 0 = f(0)$, so the only function f is the zero function $f(x) = 0$.

Particular Solutions

- Some questions don't ask for all functions satisfying some property, which often requires a lot of justification, they instead ask for just one example of such a function satisfying a property. In these cases it can help to think intuitively about the properties of satisfactory functions, and often simply making an educated guess about a type of function can solve the problem.
- Important properties to remember are ideas about logarithm rules, exponent rules, linearity, and polynomial properties.
- With that being said, particular solutions can sometimes help us find *all* solutions to a functional equation by substituting $f(x) = h(x) + p(x)$, where $p(x)$ is our particular solution, and solving for $h(x)$ (try problem 4 on the sheet).

Particular Solutions Example

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- Find a function $f : \mathbb{R} \to \mathbb{R}, f(x+y)^y = f(y)^{x+y}$.

Particular Solutions Example Solution

- Find a function $f : \mathbb{R} \to \mathbb{R}, f(x+y)^y = f(y)^{x+y}$.
- We can notice a symmetry between swapping the input of the function and the exponent. In this case since exponents distribute multiplicatively (mm words), we may guess $f(x) = e^x$ may be a solution.
- Checking, we find the left hand side is e^{xy+y^2}, and the right e^{yx+y^2}, which are equivalent and the problem is solved.

Symmetry

Looks for points in functional equations with and without symmetry between variables, then try and write two (or more) equations

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Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for all real x, y,

$$f(f(x) + f(y)) = f(f(x)) + f(x)y^2 + f(f(y)).$$

Symmetry

Looks for points in functional equations with and without symmetry between variables, then try and write two (or more) equations

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Excluding the middle RHS term, everything is symmetric: let's swap variables:

$$f(f(x) + f(y)) = f(f(x)) + f(x)y^2 + f(f(y))$$

$$f(f(x) + f(y)) = f(f(y)) + f(y)x^2 + f(f(x))$$

Subtracting the equations, we get $f(x)y^2 = f(y)x^2$. Substitute $y = 1$ and we obtain $f(x) = f(1)x^2$. Substitute $f(x) = Ax^2$ back into our equation (with $y = 0$) to find only $A = 0$ and $A = 1$ work.

A slightly harder problem

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \neq 0$ for $x \neq 0$ and, for all real x, y:

$$f(x^2) + yf(y) = f(f(x+y)^2 - 2xf(y)).$$

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Ew. Let's try a substitution! $y = 0$:

$$f(x^2) = f(f(x)^2 - 2xf(0)).$$

Let's get rid of those enclosing $f's$!

$$x^2 = f(x)^2 - 2xf(0).$$

And now substitute back in to test which functions work (exercise for the reader cause I can't be bothered solving)!

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Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \neq 0$ for $x \neq 0$ and, for all real x, y:

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What did I do wrong?

A slightly harder problem

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Let's get rid of those enclosing $f's$!

$$x^2 = f(x)^2 - 2xf(0).$$

And now substitute back in to test which functions work (exercise for the reader cause I can't be bothered solving)!

What did I do wrong? I assumed $f(a) = f(b) \implies a = b$

Injectivity and Surjectivity

Injectivity: Each input takes on a unique value

$$f(a) = f(b) \implies a = b$$

Surjectivity: Each codomain in the output is obtained

What's the use case?

Injectivity and Surjectivity

Injectivity: Each input takes on a unique value

$$f(a) = f(b) \implies a = b$$

Surjectivity: Each codomain in the output is obtained

What's the use case?
Injectivity allows us to get rid of enclosing f's
Surjectivity allows us to substitute $f(x)$ with x.

Back to our problem

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \neq 0$ for $x \neq 0$ and, for all real x, y:

$$f(x^2) + yf(y) = f(f(x+y)^2 - 2xf(y)).$$

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Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \neq 0$ for $x \neq 0$ and, for all real x, y:

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Notice we may let $x = 0$ and "vary" y.

Back to our problem

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Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \neq 0$ for $x \neq 0$ and, for all real x, y:

$$f(x^2) + yf(y) = f(f(x+y)^2 - 2xf(y)).$$

Notice we may let $x = 0$ and "vary" y.
i.e. Assume $f(a) = f(b)$. Then substitute $y = a$ and then $y = b$:

$$f(0) + af(a) = f(f(a)^2 - 2 \cdot 0 \cdot f(a))$$

$$f(0) + bf(a) = f(f(a)^2 - 2 \cdot 0 \cdot f(a))$$

Same terms cancel out, so we are left with $a = b$ (if $f(a) = f(b) = 0, a = b = 0$).
Now our mistake is fixed!

Cyclic Functions

- We may have noticed that a common theme in substitution is making multiple substitutions and then solving different equations simultaneously.
- One particular case where this is helpful is where so-called cyclic functions are present.
- These are functions where repeated applications cycle through a set of values, i.e.
- $f(f(f(x))) = x$ is a cyclic function of order 3, because applying $f(x)$ three times cycles through the values $x, f(x), f(f(x))$ before coming back to x.
- Common examples of such functions include $1/x$ and $1 - x$ of order 2, and $\frac{1}{1-x}$ and $1 - \frac{1}{x}$ of order 3.
- To solve some problems involving cyclic functions, we can just repeatedly feed the cyclic function back into itself until we have enough simultaneous equations to solve the problem.

Cyclic Functions Example

- Find all $f : \mathbb{R} - \{0, 1\} \to \mathbb{R}, f(x) + f(\frac{1}{1-x}) = \frac{1}{x}$.

Cyclic Functions Example Solution

- Find all $f : \mathbb{R} - \{0, 1\} \to \mathbb{R}, f(x) + f(\frac{1}{1-x}) = \frac{1}{x}$.
- In this case, we want to use the cyclic property of $\frac{1}{1-x}$ in our substitution.
- We get $f(\frac{1}{1-x}) + f(\frac{1-x}{-x}) = 1 - x$.
- Substituting again we get:
- $f(\frac{1-x}{-x}) + f(x) = 1 - \frac{1}{1-x}$.
- We can now solve simultaneously for $f(x)$.

Cyclic Functions Example Solution

- Find all $f : \mathbb{R} - \{0, 1\} \to \mathbb{R}, f(x) + f(\frac{1}{1-x}) = \frac{1}{x}$.
- $f(\frac{1}{1-x}) + f(\frac{1-x}{-x}) = 1 - x$. $f(\frac{1-x}{-x}) + f(x) = 1 - \frac{1}{1-x}$.
- We get $\frac{1}{x} - f(x) + f(\frac{1-x}{-x}) = 1 - x$ from the first two equations.
- From this and the third we get $1 - x - \frac{1}{x} + f(x) + f(x) = 1 - \frac{1}{1-x}$.
- Thus, we have $f(x) = \frac{1}{2}(x + \frac{1}{x} - \frac{1}{1-x})$.

A recap of techniques

- Substitute, and don't stop substituting!
- Look closely at the domain and codomain. They might restrict substitutions or function values.
- For one variable functional equations, look for cyclic functions or "order-reducing" properties
- For integer/rational domains, try expressing functions in terms of constant offsets (e.g. $f(x) = f(x-1)$).
- For getting rid of second and higher order terms (e.g. $f(f(x))$), try substituting $f(x)$, proving injectivity/surjectivity or exploiting symmetry (the last question on the sheet uses all of these!).
- Don't tunnel vision on "lower-order" functional equations - they may be too weak!

Continuity

- An important property of functions we may specify is that they be continuous.
- Recall that continuity of a function means that the limit of the function at any point is equal to the value of the function at that point.
- A common use of this is the extension of functional equations from $\mathbb{Q}$ to $\mathbb{R}$.

Continuity Example

- Find all continuous functions $f : \mathbb{R} \to \mathbb{R}, f(x + y) = f(x) + f(y)$.

Continuity Example Solution

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Arc CLUBS SUPPORTED BY Arc INDEPENDENTLY RUN

- Find all continuous functions $f : \mathbb{R} \to \mathbb{R}, f(x+y) = f(x) + f(y)$.
- This is the Cauchy functional equation again, but now in $\mathbb{R}$ rather than $\mathbb{Q}$!
- We have already solved the subproblem on the domain $\mathbb{Q} \subset \mathbb{R}$ and found that $\forall x \in \mathbb{Q}, f(x) = cx$ for some real c (c being real extends readily from our previous solution).
- Now, since any real number can be approached by arbitrarily close rationals, and since these rationals get arbitrarily closer to c times this real, by the continuity of our function we get that $f(x) = cx \ \forall x \in \mathbb{R}$ for some $c \in \mathbb{R}$.
- This can be done more carefully using real analysis, but we won't cover that here.

Attendance form :D

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Further events

Please join us for:

- Maths workshop in two weeks
- Social session on Friday
- Programming workshop next week