Combinatorics Workshop Solutions

Zac and David

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1 Problems

1. How many words can be made from the letters in "PINEAPPLE"? What if all consonants must appear together? What if no vowels may appear adjacent to each other?
   Solution:
   - 9 letters, P appears three times, E appears twice: $\frac{9!}{3! \times 2!} = 30240$
   - Group all consonants ($PNPPL$) together as a single "letter". To rearrange $(PNPPL)IEAE$, we have 5 letters and two duplicates of E, giving $\frac{5!}{2!}$ permutations. We can also rearrange $PNPPL$ in $\frac{5!}{3!}$ ways, giving us $\frac{(5!)^2}{3! \times 2!} = 1200$
   - First, treat all vowels as non-distinguishable (and similarly the consonants). We place the 4 vowels and consider the 5 "slots" around them. One consonant must be placed in each of the 3 "slots" in between vowels for them to be separated, leaving 2 consonants to be placed in 5 slots, giving us $\binom{5}{2}$ choices. Finally, we have $\frac{4!}{2!}$ ways of ordering the vowels and $\frac{5!}{3!}$ ways of ordering the consonants, giving us $\frac{5!}{3!} \times \frac{4!}{2!} \times \binom{5}{2} = 2400$

2. Find a closed form for the trinomial coefficients, that is the coefficient of $a^i b^{j-i} c^{n-j}$ in the expansion of $(a+b+c)^n$. How about the n-nomial coefficient?
   Solution: From the expansion $(a+b+c)(a+b+c)\cdots(a+b+c)$, we must multiply i values of a, $j-i$ values of b, and $n-j$ values of c. This is equivalent to the number of ways of reordering a string of length n containing i duplicates of a, $j-i$ duplicates of b and $n-j$ duplicates of c, which is $\frac{n!}{i!(j-i)!(n-j)!}$.

3. What is the probability that a number is a multiple of 2, 3, 5 or 7? Use this to estimate how many positive integers less than or equal to 2023 have a prime factor less than 10. What is the actual answer, and how close is it?
   Solution: First let's try and find the complement probability (the probability the number is not a multiple of any of those primes). We can

do this by multiplying the probabilities that a number is not each prime (since being divisible by one prime factor is independent of being divisible by another). The probability that a number is a multiple of n is $\frac{1}{n}$, so the complement of this is $1 - \frac{1}{n}$. Multiplying these for each of the prime factors $2, 3, 5, 7$, we end up with $\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right)\left(1 - \frac{1}{7}\right) = \frac{8}{35}$. So, the probability we desire is $1 - \frac{8}{35} = \frac{27}{35}$. Estimation is left as an exercise for the reader :D.

4. How many 6-digit numbers exist such that no 3 consecutive digits are ascending nor descending?
   Solution: 611184

5. Prove that for $m < k < n$, $\binom{n}{k} = \binom{m}{0}\binom{n-m}{k} + \binom{m}{1}\binom{n-m}{k-1} + ... + \binom{m}{m}\binom{n-m}{k-m}$. Preferably, this can be done with no algebra, and purely via a counting argument.
   Solution: Consider the number of committees with k people that can be chosen from a pool of n people. One way to count this is by simply writing $\binom{n}{k}$. The other way is to split the n people into two groups and pick members from each into our committee: one of size $m < k < n$ and the other of size $n - m$. We have $m + 1$ possible options: pick 0 members from the first group and k from the second $(\binom{m}{0}\binom{n-m}{k})$, pick 1 from the first group and $k - 1$ from the second $(\binom{m}{1}\binom{n-m}{k-1})$, and so on. Since these are equivalent ways of counting how many committees of size k can be formed, we conclude $\binom{n}{k} = \binom{m}{0}\binom{n-m}{k} + \binom{m}{1}\binom{n-m}{k-1} + \cdots + \binom{m}{m}\binom{n-m}{n-m}$.

6. Let S be $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Find the number of subsets A of S such that $x \in A$ and $2x \in S \implies 2x \in A$.
   Solution: Note the requirement asserted on each subset is a "recursive" one: if x is in A, then so is every $2^k x$ that is present in S (for all integers $k > 0$). We may split A into "classes" of numbers which are "equal" up to some number of multiplications of 2: specifically, $\{1, 2, 4, 8\}$, $\{3, 6\}$, $\{5, 10\}$, $\{7\}$, $\{9\}$. From each of these sets, we must choose some integer k and pick the k largest elements (e.g. for the set $\{1, 2, 4, 8\}$, we may choose none of the elements, just the 8, the 4 and 8, the $2, 4$ and 8, or all of the elements). This gives us $5 \times 3 \times 3 \times 2 \times 2 = 180$ possible subsets.

7. (Hard Pigeonhole Principle Problem) Every point in a plane is either red, green, or blue. Prove that there exists a rectangle in the plane such that all of its vertices are the same color.
   Solution 1: Choose a 4×82 rectangular grid of points. Note that each row in the grid must have at least 2 points with the same colour by the pigeonhole principle (4 points in a row, 3 colours in total). Furthermore, note that there are more rows (82) than the number of possible colour sequences ($3^4 = 81$). Thus, by the pigeonhole principle once again, two rows must have the exact same sequence of colours. From here, we can choose two points from each row with the same corresponding x-coordinates and same colours.

Solution 2: Consider just the grid of integer points (rows are labelled by its points' x-coordinates, and columns by its points' y-coordinates). Pick an infinite subset of rows whose column 0 points all have the same colour. Now pick an infinite subset of rows from this whose column 1 points all have the same colour. Repeat this for columns 2 and 3, and note that the subset of rows we have (denote R) must have at least two columns with points all of the same colour by the pigeonhole principle (4 columns, 3 possible colours). Now we simply choose 2 points from each of these two columns with the same corresponding x-coordinates to form a rectangle.

8. (IMC 2002 B2) 200 students did an exam with 6 questions. Every question was correctly answered by at least 120 students. Show that there must be two students such that every question was correctly answered by at least one of them.

   Solution: There were $120 \times 6 = 720$ correct answers in total. By the pigeonhole principle, if we label students as pigeonholes and correct answers as pigeons, at least one student must have gotten $\lceil \frac{720}{200} \rceil = 4$ correct answers. Now, consider this student. If they got no answers wrong, we may pick any other student and we are done. If they got one answer wrong, we can pick any of the 120 students who got that answer right. If they got 2 wrong answers, then for these two questions, there were $120 \times 2 = 240$ total correct answers, thus out of the remaining 199 students, by the pigeonhole principle, at least one of them must've gotten all $\lceil \frac{240}{199} \rceil = 2$ of these questions correct. Thus, we pick them and our original student to cover all 6 questions with a right answer from either of them.