Combinatorics Workshop Problems

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1 Problems

1. How many words can be made from the letters in "PINEAPPLE"? What if all consonants must appear together? What if no vowels may appear adjacent to each other?

2. Find a closed form for the trinomial coefficients, that is the coefficient of $a^i b^{j-i} c^{n-j}$ in the expansion of $(a+b+c)^n$. How about the n-nomial coefficient?

3. What is the probability that a number is a multiple of 2, 3, 5 or 7? Use this to estimate how many positive integers less than or equal to 2023 have a prime factor less than 10. What is the actual answer, and how close is it?

4. How many 6-digit numbers exist such that no 3 consecutive digits are ascending nor descending?

5. Prove that for $m < k < n$, $\binom{n}{k} = \binom{m}{0}\binom{n-m}{k} + \binom{m}{1}\binom{n-m}{k-1} + ... + \binom{m}{m}\binom{n-m}{k-m}$. Preferably, this can be done with no algebra, and purely via a counting argument.

6. Let S be $\{1,2,3,4,5,6,7,8,9,10\}$. Find the number of subsets A of S such that $x \in A$ and $2x \in S \implies 2x \in A$.

7. (Hard Pigeonhole Principle Problem) Every point in a plane is either red, green, or blue. Prove that there exists a rectangle in the plane such that all of its vertices are the same color.

8. (IMC 2002 B2) 200 students did an exam with 6 questions. Every question was correctly answered by at least 120 students. Show that there must be two students such that every question was correctly answered by at least one of them.