Competitive Programming and Mathematics Society

Mathematics Workshop

Generating functions

Cyril and Smit

Table of contents

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Introduction

- Try out problems here: https://t2maths.unswcpmsoc.com/
- Integration Bee tomorrow!
- Next (programming) workshop is in flex-week
- Next mathematics workshop is the week afterwards
- Pizza (or Subway) time! Later

Attendance form

Queue Are Code

Playing with fractions

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In fact is IS counting upwards forever if you take carrying into account

Another example

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$0.0204081632653\ldots$

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Again like last time, this doesn't just LOOK like the powers of 2, this IS the powers of 2 with carrying

Last fraction (this time very epic)

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$$0.00010102030508132134559 0\ldots$$

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$0.0001010203050813213455 90\ldots$

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The stuff we learn along the way turns out to be useful for solving combinatorics problems too

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A fraction, e.g. 0.3485 is just $0 + 3 \times 0.1 + 4 \times 0.1^2 + 8 \times 0.1^3 + 5 \times 0.1^4$ or more abstractly $f(0.1)$ where $f(x) = 0 + 3x + 4x^2 + 8x^3 + 5x^4$

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The above fractions are infinitely long polynomials (also referred to as infinite power series or formal power series) with 0.1 or 0.01 as their input

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For example, we basically demonstrated how $1/49 = f(0.01)$ where $f(x) = 2x + 4x^2 + 8x^3 + 16x^4 + ... = \sum_{n=1}^{\infty} 2^n x^n$

Geometric series

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For an infinite polynomial, convergence only occurs when $|x| < 1$. Here, $x^{n+1} \to 0$, hence the infinite power series can be written as:

$$\frac{1}{1 - x}$$

Generating functions

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Generating functions are ultimately ways of encapsulating (or "generating") sequences of numbers using functions. Traditionally, this a polynomial, or power series.

The above function generates a sequence of just 1's:

$$\sum_{k=0}^{\infty} 1x^k$$

Manipulating generating functions

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Integrate generating function	$\rightarrow$	Right-shift and divide each element by index

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$$\frac{x}{1-x-x^2}$$

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$$\frac{x}{1 - x - x^2}$$

Note that substituting $x = 0.01$ gives $\frac{100}{9899}$. Look familiar???

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Brute force approach: repeatedly differentiate and evaluate at 0.

$$G(x) = c_0 + c_1 x_1 + \cdots + c_{n-1}x^{n-1} \leftarrow \text{ killed by differentiation}$$

$$+c_n x^n$$

$$\text{killed by evaluation} \rightarrow +c_{n+1}x^{n+1} + \cdots$$

Newton's generalised binomial theorem

We define $(x)_k = x \cdot (x-1) \cdots (x-k+1)$

e.g. $(-4)_4 = (-4)(-5)(-6)(-7) = 840$

Then we extend $\binom{n}{k} = \frac{(n)_k}{k!}$ and use it in the binomial theorem ($(x+y)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k y^{n-k}$)

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$$
\begin{aligned}
(x+1)^{-3} &= \binom{-3}{0} + \binom{-3}{1} x^1 + \binom{-3}{2} x^2 + \cdots \\
&= \binom{2}{0} - \binom{3}{1} x^1 + \binom{4}{2} x^2 - \cdots
\end{aligned}
$$

This is hard to motivate and see the applicability of without a problem.

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Stars and bars method: consider n "stars" and 2 "bars" - regions between bars are categories, stars are sandwiches: $* * * * | * * * * * * | * * * *$. Out of $n+2$ positions, 2 must be chosen for the bars, so $\binom{n+2}{2}$.

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$$=\sum_{k=0}^{\infty}\binom{-3}{k}(-x)^k=\sum_{k=0}^{\infty}\binom{k+2}{2}x^k.$$

The number of ways to choose n sandwiches is the x^n coefficient.
How does this generalise (the formula for k different flavours is $\binom{n+k-1}{k}$ btw).

How Dicey

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x^{24} is only obtainable by multiplying the x^4, x^0 and x^{20} terms or the x^4, x^{12} and x^8 terms (respective to each factor:

$$1 \times 1 \times \binom{23}{20} + 1 \times -4 \times \binom{11}{8} = 1111$$

Some more ideas to consider

- Differentiation/integration to extract coefficients
- Evaluating at roots of unity (filters out x^{kn} coefficients)
- Partial fraction decomposition
- Substitute fractional powers for the generalised binomial theorem

Further events

Please join us for:

- Integration bee tomorrow!
- Rookie Code Rumble right now!
- T2 Mathematics Competition right now!

Feedback form