Competitive Programming and Mathematics Society

Mathematics Workshop

Inequalities

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Welcome

- Programming workshop next week
- And that's a wrap! I think
- Try some problems: https://t2maths.unswcpmsoc.com/
- Slides will be uploaded on website (unswcpmsoc.com)

Attendance form

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Something different...

Inequalities can be a little tricky, so the first few problems on the problem sheet are marked with what inequality you should consider using!

Square numbers are non-negative!

- The most important inequality in maths is that the square of all real numbers is always greater than or equal to 0.
- We can express this as $x^2 \geq 0$ where x is any real number.
- This idea extends to mathematical expressions as well. For example, the expressions a^4 and $(a+b)^2 = a^2 + 2ab + b^2$ are both greater than or equal to 0 for any $a, b \in \mathbb{R}$.
- **A number squared is 0 if and only if the number itself is 0.**
- **Furthermore,** $a^2 \geq b^2$ **if and only if** $|a| \geq |b|$**.**
- Problem: Prove that $a^2 + b^2 \geq 2ab$ for all $a, b \in \mathbb{R}$.

Square numbers - Solution

- Problem: Prove that $a^2 + b^2 \geq 2ab$ for all $a, b \in \mathbb{R}$.
- Working backwards, this is logically the same as proving that $a^2 + b^2 - 2ab \geq 0$.
- Because $(a - b)^2 = a^2 + b^2 - 2ab$ and because $(a - b)^2$ is the square of a real number, we can conclude that $a^2 + b^2 - 2ab \geq 0$.

AM-GM inequality

- One of the most famous (and widely used) inequality theorems is the **Arithmetic Mean-Geometric Mean inequality**.
- It states that, for any set of two or more non-negative real numbers (denoted as $a_1, a_2, ..a_n$), that their arithmetic mean (average) is greater than their geometric mean.
- This can be expressed as $\frac{a_1+a_2+...+a_n}{n} \geq \sqrt[n]{a_1 a_2 ... a_n}$.
- **Equality holds if and only if $a_1 = a_2 = ... = a_n$.**
- Problem: Prove that the AM-GM inequality holds true for any two non-negative real numbers.

AM-GM inequality - Proof

- Problem: Prove that the AM-GM inequality holds true for any two non-negative real numbers.
- Let us denote these numbers as x and y. Our objective is to prove that $\frac{x+y}{2} \geq \sqrt{xy}$ holds true for any value of x and y. This is the same as proving that $\frac{x+y-2\sqrt{xy}}{2} \geq 0$.
- Notice that the numerator of the expression on the left hand side is equal to $(\sqrt{x} - \sqrt{y})^2$.
- Since square numbers are non-negative, therefore $\frac{x+y-2\sqrt{xy}}{2} = \frac{(\sqrt{x}-\sqrt{y})^2}{2} \geq 0$.

AM-GM inequality - Proof

- Problem: Prove that the AM-GM inequality holds true for any two non-negative real numbers.
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- Notice that the numerator of the expression on the left hand side is equal to $(\sqrt{x} - \sqrt{y})^2$.
- Since square numbers are non-negative, therefore $\frac{x+y-2\sqrt{xy}}{2} = \frac{(\sqrt{x}-\sqrt{y})^2}{2} \geq 0$.
- There's multiple ways to prove the AM-GM inequality for more numbers, such as by using induction, logarithm approximations or other inequalities.

AM-GM - Splitting

- You have to be creative with what you use as your set of numbers!
- Problem: For non-negative numbers $x, y, z \in \mathbb{R}$ such that $x + y + z = 1$, find the maximum value of xy^2z^3. Also find the values of x, y, z such that this maximum is achieved.

AM-GM - Solution

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- Problem: For non-negative numbers $x, y, z \in \mathbb{R}$ such that $x + y + z = 1$, find the maximum value of xy^2z^3. Also find the values of x, y, z such that this maximum is achieved.
- If we use AM-GM straight away, we will get the inequality $\sqrt[3]{xyz} \leq \frac{x+y+z}{3} = \frac{1}{3}$. Whilst this inequality isn't wrong, it's not what we are looking for.
- Instead, rewrite $x + y + z = 1$ as $x + \frac{y}{2} + \frac{y}{2} + \frac{z}{3} + \frac{z}{3} + \frac{z}{3} = 1$. Now, by applying AM-GM, we get $\frac{1}{6} = \frac{x+\frac{y}{2}+\frac{y}{2}+\frac{z}{3}+\frac{z}{3}+\frac{z}{3}}{6} \geq \sqrt[6]{x\frac{y}{2}\frac{y}{2}\frac{z}{3}\frac{z}{3}\frac{z}{3}} = \sqrt[6]{\frac{xy^2z^3}{108}}$. Rearranging this, we get $xy^2z^3 \leq \frac{1}{432}$.
- Equality holds if and only if all the terms are equal. So if we equate $x = \frac{y}{2} = \frac{z}{3}$, we'd find that the maximum is achieved when $x = \frac{1}{6}, y = \frac{1}{3}, z = \frac{1}{2}$.

Triangle substitution - What is it?

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- Occasionally, you would find an inequality with three variables (let us call them a, b, c) and a condition which states that these three variables form the sides of a triangle.

Triangle substitution - What is it?

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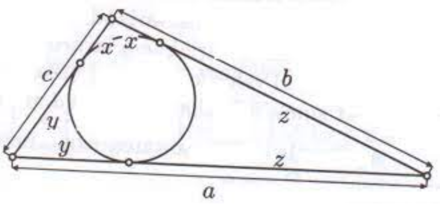
- Every triangle has an incircle (a circle which is tangent to all three edges of the triangle). Notice that on this diagram, we can substitute $a = y + z, b = z + x, c = x + y$.
- To swap from x, y, z back to a, b, c, we use the substitutions $x = \frac{b+c-a}{2}, y = \frac{c+a-b}{2}, z = \frac{a+b-c}{2}$. **An easy way to remember this is that a is opposite to x, b is opposite to y and c is opposite to z.**



Triangle substitution - Problem

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- Problem: Let a, b, c be the sides of a triangle. Prove that $abc \geq (b+c-a)(c+a-b)(a+b-c)$.

Triangle substitution - Problem

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- Problem: Let a, b, c be the sides of a triangle. Prove that $abc \geq (b+c-a)(c+a-b)(a+b-c)$.
- We use the triangle substitutions $a = y+z, b = z+x, c = z+x$. Expanding out the left hand side, we get

$$\begin{aligned}
abc = (x+y)(y+z)(z+x) &= (x^2y + yz^2) + (y^2z + zx^2) + (z^2x + xy^2) + 2xyz \\
&\geq 2\sqrt{x^2y^2z^2} + 2\sqrt{x^2y^2z^2} + 2\sqrt{x^2y^2z^2} + 2xyz \\
&= 8xyz \\
&= 8\frac{b+c-a}{2}\frac{c+a-b}{2}\frac{a+b-c}{2} \\
&= (b+c-a)(c+a-b)(a+b-c).
\end{aligned}$$

Cauchy-Schwartz inequality

- From high school, you may remember the dot product:

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \cdot \begin{pmatrix}5\\3\\4\end{pmatrix} = 1 \cdot 5 + 2 \cdot 3 + 3 \cdot 4$$

- If the vectors are $\vec{u}, \vec{v}$, then it turns out $\vec{u} \cdot \vec{v} = |\vec{v}||\vec{u}| \cos\theta$
- Since $|\cos\theta| \leq 1$, $u \cdot v \leq |\vec{u}||\vec{v}|$

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- This is the Cauchy-Schwartz inequality! Also written as

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

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- Tip: look for squares/force square terms to use Cauchy-Schwartz

Cauchy-Schwartz inequality - an example

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Show that for all positive real numbers a, b, c such that $abc = 1$

$$\frac{1}{c^3(a+b)} + \frac{1}{b^3(a+c)} + \frac{1}{a^3(b+c)} \geq \frac{3}{2}$$

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Arc CLUBS SUPPORTED BY Arc INDEPENDENTLY RUN

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Something tempting is to use Cauchy-Schwartz inequality backwards (since this looks like a dot product), however this will give us an $\leq$. Notice we can isolate square terms $\frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2}$, so maybe we consider

$$\left(\frac{1}{c^2\sqrt{c(a+b)}^2} + \frac{1}{b^2\sqrt{b(a+c)}^2} + \frac{1}{a\sqrt{a(b+c)}^2}\right)\left(\sqrt{c(a+b)}^2 + \sqrt{b(a+c)}^2 + \sqrt{a(b+c)}^2\right)$$

$$\geq \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2$$

Then we divide the right term of the LHS on both sides to obtain

$$\frac{1}{c^3(a+b)} + \frac{1}{b^3(a+c)} + \frac{1}{a^3(b+c)} \geq \frac{\frac{(ab+bc+ca)^2}{(abc)^2}}{2(ab+bc+ca)} = \frac{ab+bc+ca}{2} \geq \frac{3\sqrt[3]{(abc)^2}}{2} = \frac{3}{2}$$

Jensen's inequality

- Say we have a function f, and some values $x_1, x_2, \ldots$.
- Which is bigger, f(the average of $x_1, x_2, \ldots$), or the average of $f(x_1), f(x_2), \ldots$?

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- This depends. What if f is convex? (convex means "cupped upwards")

Jensen's inequality

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- Which is bigger, f(the average of $x_1, x_2, \ldots$), or the average of $f(x_1), f(x_2), \ldots$?
- This depends. What if f is convex? (convex means "cupped upwards")
- Jensen's inequality says evaluating then averaging gives a bigger number!
- We write $\mathbb{E}(h(X)) \geq h(\mathbb{E}(X))$ (this means weighted averages also work!)
- For a concave function, $\mathbb{E}(h(X)) \leq h(\mathbb{E}(X))$

Jensen's Inequality Example

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- Let's prove the power mean inequality
- For positive integers n, k and a set of positive real numbers $x_1, x_2, \ldots, x_n$, show that

$$\left(\frac{x_1^k + x_2^k + \cdots + x_n^k}{n}\right)^{\frac{1}{k}} \leq \left(\frac{x_1^{k+1} + x_2^{k+1} + \cdots + x_n^{k+1}}{n}\right)^{\frac{1}{k+1}}$$

- Note it looks like we have averages on both sides... what convex (or concave) function could we consider?

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- Note it looks like we have averages on both sides... what convex (or concave) function could we consider?
- Since we want something with a $k+1$ and a k, let's try $h(x) = x^{\frac{k+1}{k}}$, and apply it on $x_1^k, x_2^k, \ldots$:
-

$$\left(\frac{x_1^k + x_2^k + \cdots + x_n^k}{n}\right)^{\frac{k+1}{k}} \leq \left(\frac{x_1^{k \cdot \frac{k+1}{k}} + x_2^{k \cdot \frac{k+1}{k}} + \cdots + x_n^{k \cdot \frac{k+1}{k}}}{n}\right)$$

Rearrangement inequality

- Given increasing sequences of *any* real numbers $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots b_n$, $a_1b_1 + a_2b_2 + \cdots + a_nb_n \geq a_1b_{\sigma(1)} + a_2b_{\sigma(2)} + \cdots + a_nb_{\sigma(n)} \geq a_1b_n + a_2b_{n-1} + \cdots + a_nb_1$, where σ is some permutation function which rearranges the numbers 1 to n.

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- Roughly, this means the maximum dot product we can achieve between two vectors is if every kth highest number from one vector is paired with the kth highest number from the other vector.

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- Roughly, this means the maximum dot product we can achieve between two vectors is if every kth highest number from one vector is paired with the kth highest number from the other vector.
- Similarly, the minimum dot product is achived when the kth highest from one is paired with the kth *lowest* from the other.

Rearrangement inequality - example

Prove for positive real numbers a, b, c,

$$\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab} \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

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WLOG assume $a \geq b \geq c$. Then $ab \geq ca \geq bc$ and $\frac{1}{ab} \leq \frac{1}{ca} \leq \frac{1}{bc}$. This means the "maximal pairing" is a, b, c with $\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ (respectively), which is the LHS.

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$$\frac{a}{ab} + \frac{b}{bc} + \frac{c}{ca} = \frac{1}{b} + \frac{1}{c} + \frac{1}{a}$$

Fun(ny) exercise

Try proving Cauchy with Jensen and Rearrangement (separately)!

An interesting observation...

A lot of the inequalities we've seen so far have a peculiar property: the multinomial terms on both sides all seem to have the same "degree"

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A lot of the inequalities we've seen so far have a peculiar property: the multinomial terms on both sides all seem to have the same "degree"

Homogenisation

- Well, I define "degree" of "multinomial terms" here very weirdly
- Replace every variable with "x", separate the added terms and check the values of the exponents
- All terms seem to have the same degrees -> AM-GM, Cauchy-Schwartz, Rearrangement, and standard operations of multiplying and adding seem to retain "degree"
- Inequalities with different degree terms tend to have constraints imposed (e.g. $abc = 1$)
- You can force inequalities to have the same degree on both sides by using the constraint $\rightarrow$ this process is called "homogenisation"

Homogenisation - an example

Show that for positive real numbers a, b, c where $abc = 1$

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq a + b + c$$

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To homogenise, we divide by $\sqrt[3]{abc}$ on the RHS (get creative when trying to homogenise!)

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The cube root indicates the use of AM-GM with three variables.

Homogenisation - an example

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To homogenise, we divide by $\sqrt[3]{abc}$ on the RHS (get creative when trying to homogenise!) The cube root indicates the use of AM-GM with three variables. Is there a cyclic inequality that gives us $\frac{a}{\sqrt[3]{abc}}$?

$$\frac{a}{b} + \frac{a}{b} + \frac{b}{c} \geq 3\sqrt[3]{\frac{a^2}{bc}} = \frac{3a}{\sqrt[3]{abc}}$$

Now just add the cyclic inequalities up.

Attendance form :D

Further events

Please join us for:

- Maths workshop in two weeks
- Social session on Friday
- Programming workshop next week