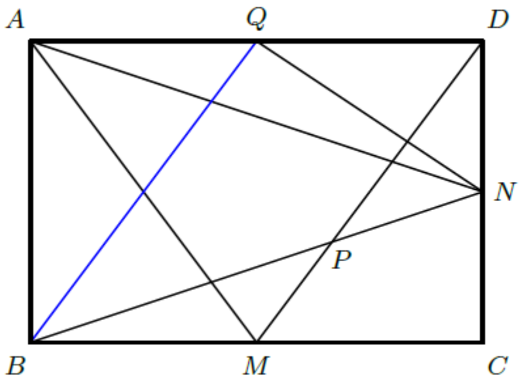
PROBLEM SET 2

1 We use the symmetry of the rectangle to redraw the situation near A, over near B. That is, let Q be the midpoint of AD and construct BQ. Then, by symmetry, $\angle QBN = \angle MAN$.

All that we need to prove now is that BQ and MD are parallel. But once we observe that BM and QD are equal and parallel, it follows that $BQDM$ is a parallelogram and so BQ is parallel to MD as desired.

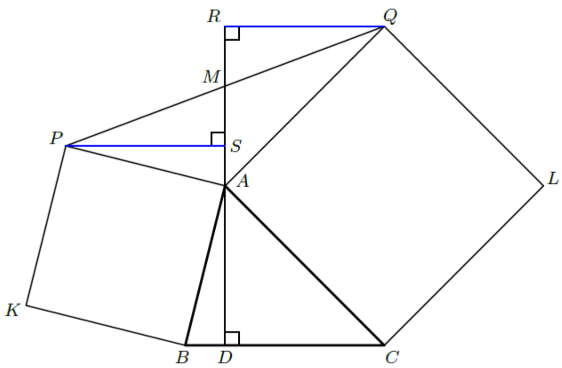


2 First, let's do a bit of angle chasing. If we let $\angle CAD = \alpha and \angle BAD = \beta$, then we have $\angle ACD = 90° - \alpha$ and $\angle ABD = 90° - \beta$. Also, since angles on a straight line add to 180°, we have $\angle QAM = 90 - \alpha$ and $\angle PAM = 90 - \beta$.

There are several ways to proceed from here, but one way is as follows. Note that $QA = AC$ and $\angle QAM = \angle ACD$. These are quite similar situations, and by drawing a single line segment, we can create a pair of congruent triangles.

So let R be the foot of the perpendicular from Q to the line MD. This gives us the congruent triangles QAR and ACD, as desired. But what we have done on the right side of the diagram, we can similarly do on the left. So let S be the foot of the perpendicular from P to the line MD, so that we have congruent triangles PAS and ABD. In particular, we have managed to prove that $QR = AD = PS$.

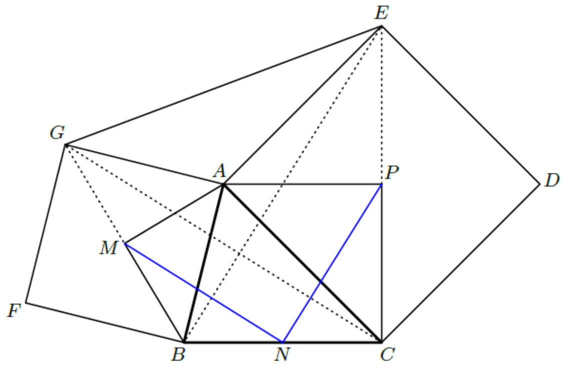
Therefore, M is horizontally halfway between P and Q. (can be proven with congruence) This means that M must be the midpoint of PQ, and we are done.



3 Construct squares $ABFG$ and $ACDE$ as shown.

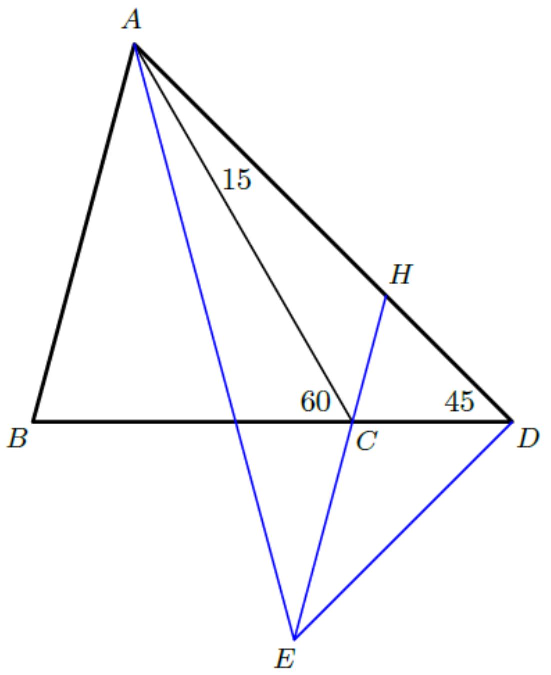
We see that M and P are the centres of these two squares. In fact, since $BM = \frac{1}{2}BG$ and $BN = \frac{1}{2}BC$, triangles BMN and BGC are similar. So, $MN = \frac{1}{2}GC$, and $MN \parallel GC$. Similarly, $NP = \frac{1}{2}BE$ and $NP \parallel BE$. Thus it suffices to show that $GC = BE$ and $GC \perp BE$.

These facts are some of the nice properties of this diagram that you may already know from elsewhere. Either way, note that $\angle GAC = \angle GAB + \angle BAC = \angle EAC + \angle BAC = \angle BAE$ and $GA = BA$ and $AC = AE$. Therefore, triangles GAC and BAE are congruent. (In fact one is a rotation of the other about A by $90°$.) It follows that $GC = BE$ and $GC \perp BE$ as required.



4 Since $DC : CB = 1 : 2$, it might be worthwhile to make a construction so that C will be a centroid of some triangle. But that does not seem to work, and so we will instead try to make C the incentre of a triangle. Incentres, being the intersection of angle bisectors, always give us lots of information about angles, and information about ratios, which is what we want. So, paint a picture where C is the incentre of some triangle. A little angle chasing gives $\angle DAC = 15°$. So construct a point E such that $\angle DAE = 30°$ and $\angle ADE = 90°$. Then C is the incentre of triangle AED. Furthermore, AED is a $30° - 60° - 90°$ triangle. Label points as shown with H being the intersection of EC and AD. Note that EHD is also a $30° - 60° - 90°$ triangle.

Diagram is in the next page.



By the angle bisector theorem we have

$$\frac{AH}{HD} = \frac{AE}{DE} = 2.$$

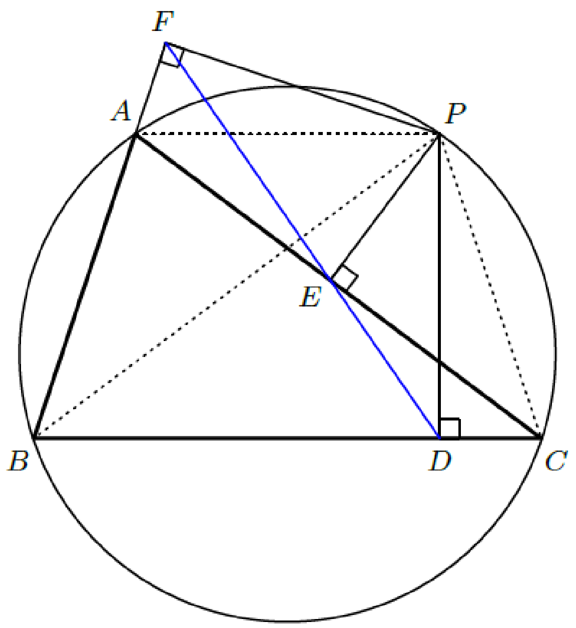
We were also given

$$\frac{BC}{CD} = 2.$$

Hence triangles HCD and ABD are similar and therefore, $\angle BAD = 60^\circ$.

5 Since BAF is a straight line, it suffices to show that $\angle BFD = \angle AFE$. From cyclic quadrilaterals $PFBD$ and $PFAE$ we know that $\angle BFD = \angle BPD$ and $\angle AFE = \angle APE$. If we subtract $\angle BPE$ from both these angles, then it suffices to prove that $\angle EPD = \angle APB$. However, both these angles are equal to $\angle BCA$ - the first is because $CDEP$ is cyclic, and the second is because $BAPC$ is cyclic.

This solution is not complete since we have not addressed issues of diagram dependence. We leave this for the reader to complete.

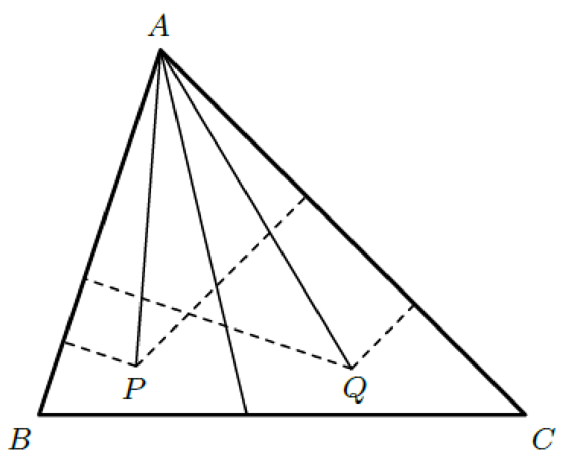


6 Let p_a, p_b and p_c denote the distances from P to the lines BC, CA and AB, respectively. Similarly, for any point Q in the plane of triangle ABC, we let q_a, q_b and q_c denote the distances from Q to the lines BC, CA and AB, respectively.

Observe that Q lies on the line obtained by reflecting the line PA through the angle bisector at A if and only if

$$\frac{q_b}{q_c} = \frac{p_c}{p_b}$$

obtained thru similar triangle.



Next, define Q to be the point which lies on the reflected line through A and the reflected line through B.

Therefore, the previous observation gives us the two equations

$$\frac{q_b}{q_c} = \frac{p_c}{p_b} \text{ and } \frac{q_c}{q_a} = \frac{p_a}{p_c}$$

Multiplying these two equations together, we obtain

$$\frac{q_b}{q_a} = \frac{p_a}{p_b}.$$