Competitive Programming and Mathematics Society

Geometry

Workshop 2, Week 5, Term 3, 2021

CPMSoc Mathematics

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Angle chasing

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Draw them out!

Alternate segment theorem

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Theorem (Alternate segment theorem)

Let A, B, C be points on a circle, and let PA be a line segment such that P lies on the opposite side of line AB as C. Then the line PA is tangent to the circle at A if and only if $\angle ACB = \angle PAB$.

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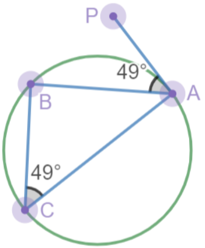

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Example

In parallelogram $ABCD$, AC is longer than BD. Let P be a point on AC such that $BCDP$ is a cyclic quadrilateral.
Prove that BD is a common tangent to the circumcircles of triangle ADP and triangle ABP.

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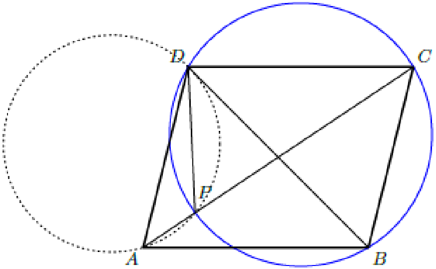

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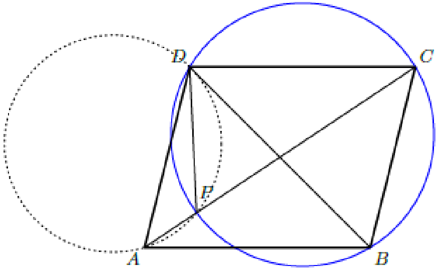
By the alternate segment theorem, it is sufficient to prove that $\angle PDB = \angle DAP$ and $\angle PBD = \angle BAP$.



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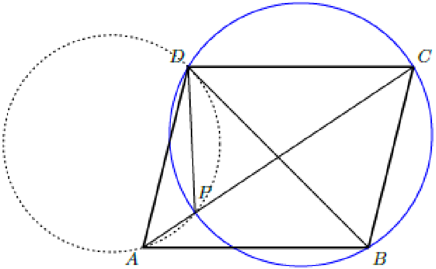
Since the quadrilateral $BCDP$ is cyclic, we have $\angle PDB = \angle PCB$.



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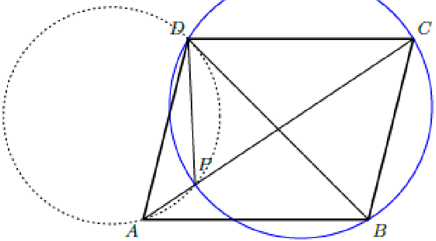
We can also deduce that $\angle PCB = \angle DAP$ because AD and BC are parallel.



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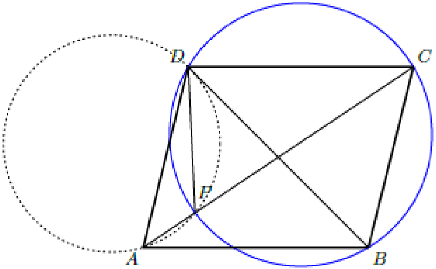

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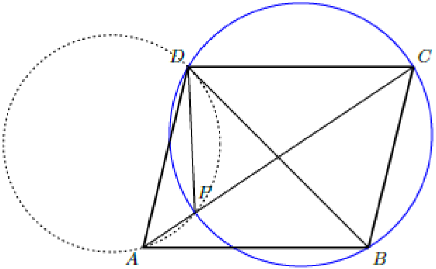
The other equality can be proven by an analogous argument.



Cyclic quadrilaterals

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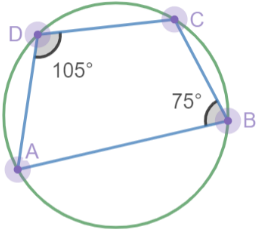

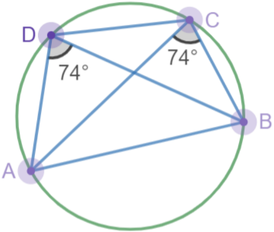

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In triangle ABC, points D and E are located on the side BC such that AD is an altitude and AE is an angle bisector. The point M on AE is such that BM is perpendicular to AE and the point N on AC is such that EN is perpendicular to AC.
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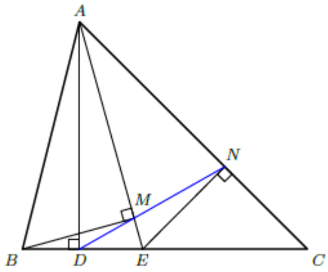

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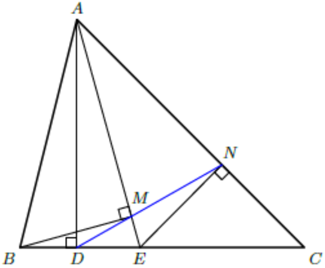
From the cyclic quadrilateral theorems, we know that $ABDM$ is cyclic because $\angle ADB = \angle AMB = 90°$. We also know that $ADEN$ is cyclic because $\angle ADE + \angle ANE = 90° + 90° = 180°$.



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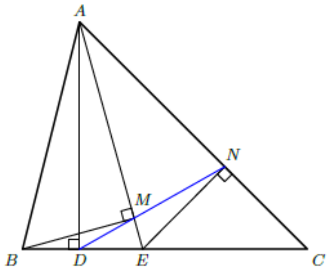

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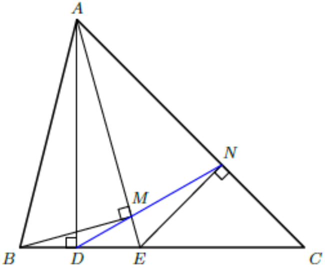
We will label $\angle BAC = 2\alpha$.
Then we use this to label as many other angles in the diagram as possible.
For a start, we have $\angle BAE = \angle CAE = \alpha$.



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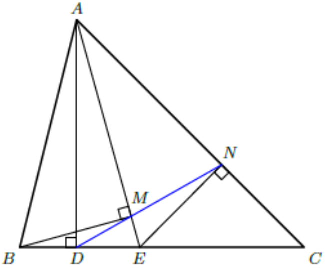
The cyclic quadrilateral $ABDM$ tells us that $\angle BDM = 180^\circ - \angle BAM = 180^\circ - \angle BAE = 180^\circ - \alpha$.



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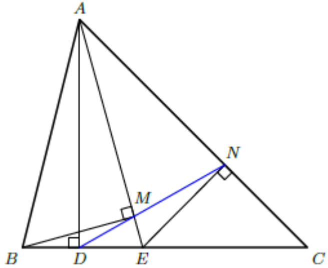

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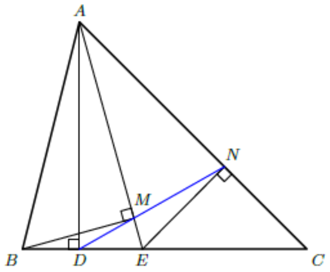
Hence, we get $\angle BDM + \angle NDC = 180°$.



More examples!

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Let $ABCD$ be a square and P be a point on its side BC. The circle passing through points A, B and P intersects BD once more at point Q. The circle passing through points C, P and Q intersects BD once more at point R.
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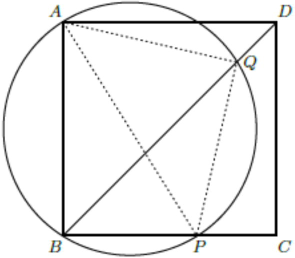

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Since BD is the diagonal
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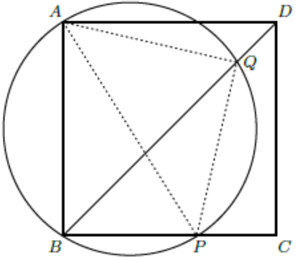

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Clearly $ABPQ$
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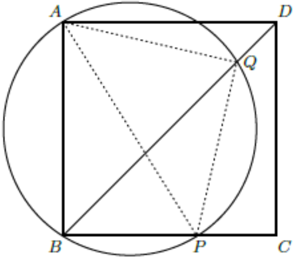
and $\angle PAQ = \angle PBQ = 45°$.



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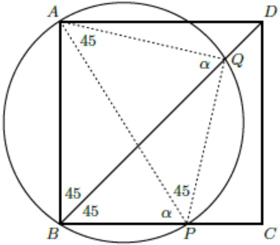
Similarly, letting $\angle APB = \alpha$.
we can chase all the angles around.



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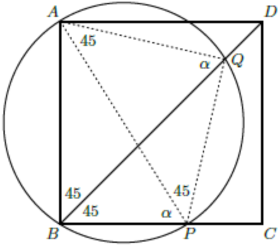
Some of the other angles in the diagram are $\angle BQP = \angle BAP = 90 - \alpha$ and $\angle QPC = 135° - \alpha$.



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Drawing CQ in, we get $AQ = CQ$
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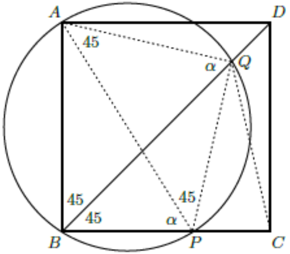

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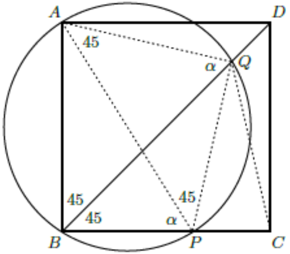
Thus,
$AQ = CQ = PQ$. So CPQ is isosceles and $\angle QCP = \angle QPC = 135° - \alpha$.



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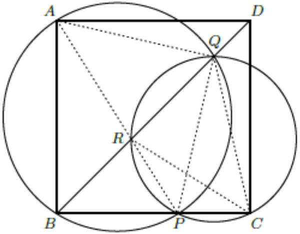
Now we'll put the second circle in.



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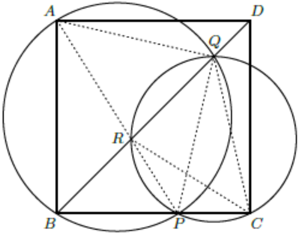
We now have $\angle PCR = \angle PQR = 90° - \alpha$. Since $\angle PCQ = 135° - \alpha$, we have $\angle RCQ = 45°$. Thus $\angle RPQ = \angle RCQ = 45°$. But now $\angle RPQ = \angle APQ = 45°$. Therefore, points A, R and P are collinear as required.



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With this in mind, we extend the line AC to the point P such that $CP = CB$. Of course, what we now need to prove is that $AD = DP$.

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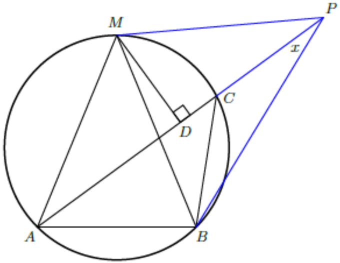
If $AD = DP$, then we would know that M lies on the perpendicular bisector of AP. Since M also lies on the perpendicular bisector of AB, it must be the case that M is the circumcentre of triangle ABP. Let's aim to prove this using an angle chase.



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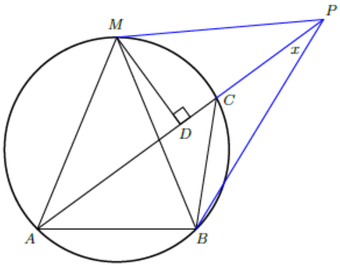

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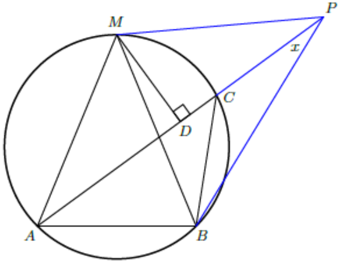
From this, it follows that $\angle ACB = 2x$ and since $ABCM$ is a cyclic quadrilateral, we also have $\angle AMB = 2x$.



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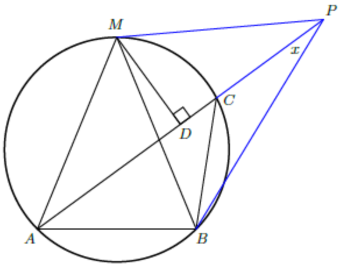
Prove that $AD = DC + CB$.

The chord AB subtends an angle $2x$ at M with $AM = BM$ and an angle x at P.[1] Since P and M lie on the same side of AB, the point M is indeed the circumcentre of triangle ABP. Therefore, MD splits the isosceles triangle AMP into two congruent triangles, so $AD = DP$.



1: The angle subtended by an arc of a circle at its center is twice the angle it subtends anywhere on the circle's circumference.

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Although there is a trigonometric approach to this problem, without trigonometry the problem is difficult to approach directly.

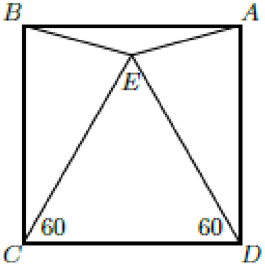
Reverse constructions

CPMSOC

Example

Point O lies inside square $ABCD$ such that $\angle OAB = \angle OBA = 15°$.
Prove that triangle ODC is equilateral.

Let E be the point inside $ABCD$ such that EDC is equilateral.



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CPMSOC

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Point O lies inside square $ABCD$ such that $\angle OAB = \angle OBA = 15°$.
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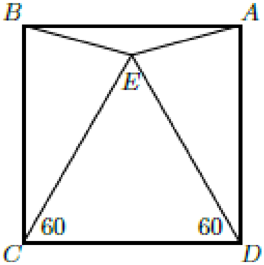
We now aim to show that $\angle EAB = \angle EBA = 15°$, so that E and O are the same point. (This follows since there is only one possible point O inside $ABCD$ satisfying the conditions $\angle OAB = \angle OBA = 15°$.)



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Point O lies inside square $ABCD$ such that $\angle OAB = \angle OBA = 15°$.
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As triangle CDE
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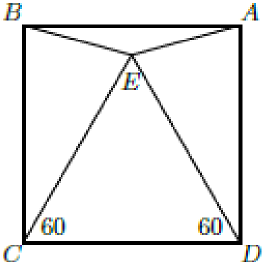
So triangle CBE is isosceles.



Reverse constructions

CPMSOC

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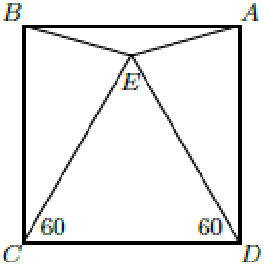
So triangle CBE is isosceles.

But since $\angle BCE = 30°$
we have $\angle CEB = \angle CBE = 75°$ and
so $\angle EBA = 15°$. Similarly, $\angle EAB = 15°$,
as desired. Therefore $O = E$
and triangle $ODC = EDC$ is equilateral.



Collinearity

As we can see from one of the previous example, one way to prove that three points A, B, C are collinear is to prove that $\angle ABC = 0^\circ$ or 180°.

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Let ABC and ADE be similar triangles whose vertices are labelled clockwise. Let P be the second common point of the circumcircles of the triangles besides A.
Show that P must lie on the line connecting B and D.

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CPMSOC

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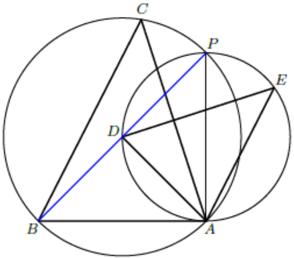
$\angle BPA = \angle BCA = \angle DEA = \angle DPA$.



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The first equality follows
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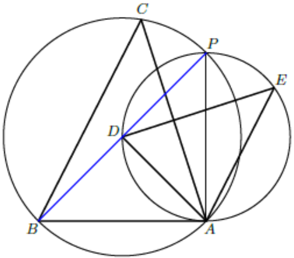

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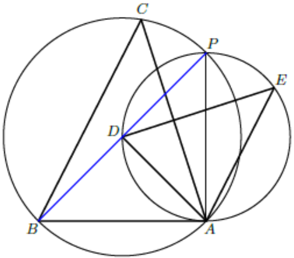

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The third follows
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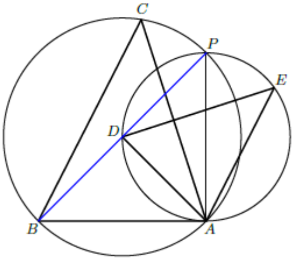

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Let ABC and ADE be similar triangles whose vertices are labelled clockwise. Let P be the second common point of the circumcircles of the triangles besides A.
Show that P must lie on the line connecting B and D.

But seeing that B and D lie on the same side of the line AP, the equality $\angle BPA = \angle DPA$ tells us that P must lie on the line passing through B and D.

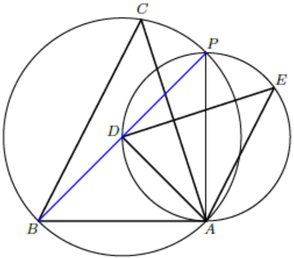
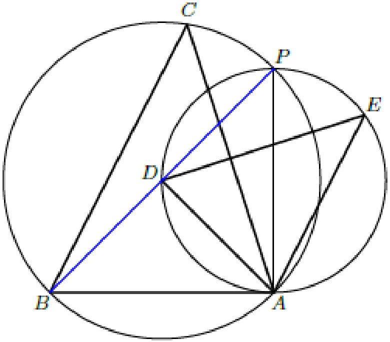


Diagram dependence

However, we are not done yet. It is time to raise a important pitfall in geometry known as diagram dependence. We only solved the problem for the diagram shown. It is possible to have other diagrams where the relative positions of the points are different, and our angle chase is a bit different. For instance, if triangle ADE were rotated clockwise until D lay on ray AP beyond P, then it is no longer true that $\angle DEA = \angle DPA$, but instead we would have $\angle DEA = 180° - \angle DPA$.



Can you identify all the different configurations possible and solve in each case?

Menelaus' theorem

Theorem (Menelaus' theorem)

If X, Y and Z lie on the three (possibly extended) sides BC, AC and AB of a triangle ABC, then the three points X, Y and Z are collinear if and only if

$$\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = -1$$

where the segments are considered to have directed length.

Menelaus' theorem

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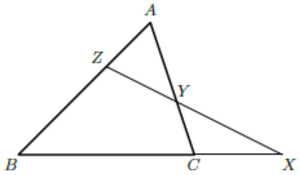
The part about directed lengths in the statement of Menelaus' theorem simply means that the ratios take into account the directions of the vectors $\vec{AZ}, \vec{ZB}$, and so forth. Thus $\frac{AZ}{ZB}$ is a positive ratio if Z lies on segment AB, and is a negative ratio otherwise.



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Suppose that ABC is a triangle with circumcircle in which the three tangents to at A, B and C meet the three opposite sides at X, Y and Z, respectively.
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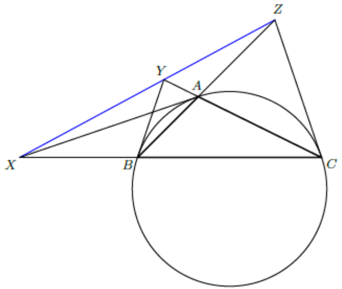

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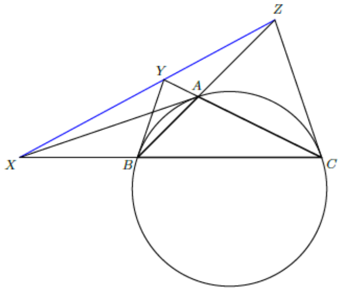
By cancelling out XA, we get $\frac{BX}{XC} = -\frac{AB^2}{AC^2}$



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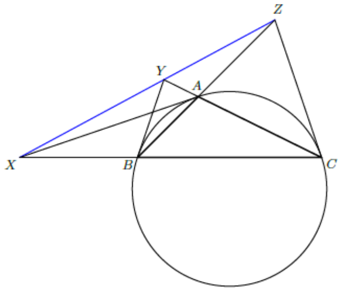

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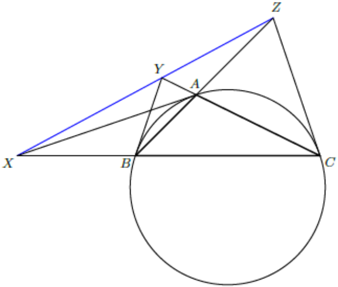
These all multiply together and cancel out to give -1.
Thus X, Y and Z are collinear by Menelaus' theorem.



Ceva's theorem

CPMSOC

Theorem (Ceva's theorem)

If X, Y and Z lie on the three (possibly extended) sides BC, AC and AB of a triangle ABC, then the three lines (called cevians) AX, BY and CZ are concurrent if and only if

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Note that this is exactly the same expression as for Menelaus' theorem except that we have +1 on the right-hand side instead of -1.

Concurrency

Example (Ceva's theorem)

Outside triangle ABC, points K, L and M are constructed in such a way that $\angle MAB = \angle LAC$; $\angle KBC = \angle MBA$ and $\angle LCA = \angle KCB$.
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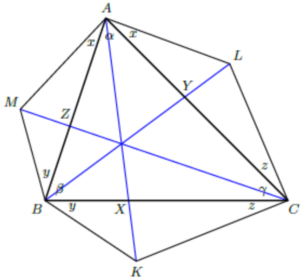
Let $x = \angle MAB$, $y = \angle KBC$ and $z = \angle LCA$ and let AK, BL and CM intersect BC, CA and AB at points X, Y and Z, respectively, as in the diagram.



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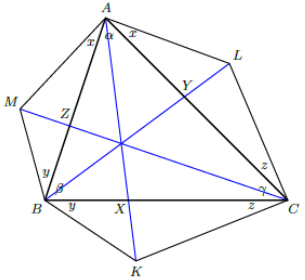
By Ceva's theorem
it would suffice to prove that $\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = +1$.



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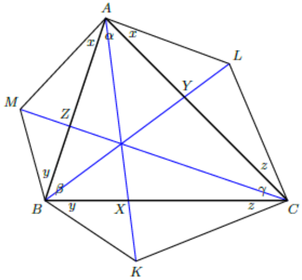
Let P denote
this product. Note that $\frac{BX}{XC} = \frac{\triangle ABX}{\triangle ACX} = \frac{\triangle KBX}{\triangle KCX}$



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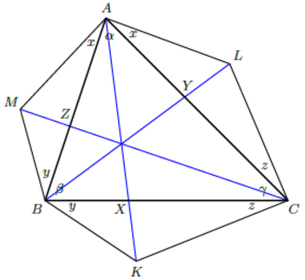
Using addendo[1] , we have

$$\frac{BX}{XC} = \frac{\triangle ABK}{\triangle ACK} = \frac{\frac{1}{2}AB \cdot BK sin(\beta + y)}{\frac{1}{2}AC \cdot CK sin(\gamma + z)}.$$



1: If $r = \frac{a}{b} = \frac{c}{d}$, then $r = \frac{a+c}{b+d}$

Ceva's theorem

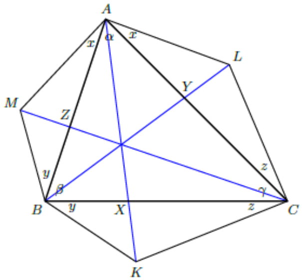
CPMSOC

Example (Ceva's theorem)

Outside triangle ABC, points K, L and M are constructed in such a way that $\angle MAB = \angle LAC$; $\angle KBC = \angle MBA$ and $\angle LCA = \angle KCB$.
Prove that the three lines AK, BL and CM are concurrent.

We obtain similar expressions for the other two ratios and thus compute after cancelling out that

$$P = \frac{KB}{KC} \cdot \frac{LC}{LA} \cdot \frac{MA}{MB}.$$



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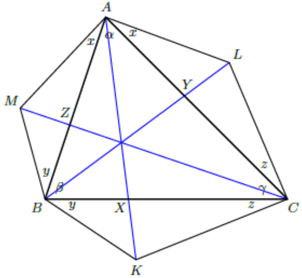
Prove that the three lines AK, BL and CM are concurrent.

Finally, we use the sine rule in triangle KBC to find that

$$\frac{KB}{KC} = \frac{sinz}{siny}.$$



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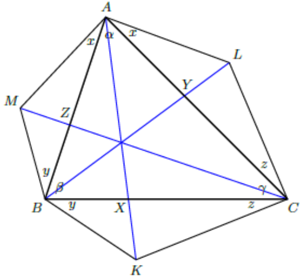
We obtain similar expressions for the other two ratios so that we finally compute that $P = +1$. Therefore, AX, BY and CZ are concurrent by Ceva's theorem.



Ceva's theorem

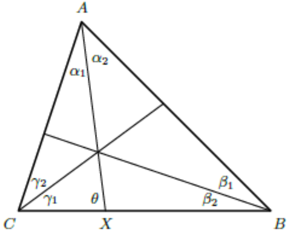
There is also a trigonometric version of Ceva's theorem.

Theorem (Ceva's theorem)

If angles are marked as in the figure, then the cevians are concurrent if and only if

$$\frac{sin\alpha_1}{sin\alpha_2}\cdot\frac{sin\beta_1}{sin\beta_2}\cdot\frac{sin\gamma_1}{sin\gamma_2}=+1$$

The proof of this is quite straightforward and
may be carried out by using the sine rule six times.



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$$\frac{CX}{sin\alpha_1} = \frac{AC}{sin\theta} \text{ and } \frac{XB}{sin\alpha_2} = \frac{AB}{sin(180^\circ - \theta)}$$

and so we obtain equations such as

$$\frac{sin\alpha_1}{sin\alpha_2} = \frac{AB}{AC} \cdot \frac{CX}{XB}.$$