PROBLEM SET 1

1 Two players start with the number 1 and take turns to multiply it by an integer from 2 to 9. The winner is the first player to obtain a number greater than or equal to 1000. Which player has a winning strategy?

2 *Chomp* is played with an $m \times n$ grid of chocolate. Two players take turns to eat a square of chocolate, along with every square which is above and to the right of it. Unfortunately the bottom-left square is poisonous, so that the player who is forced to eat it is considered the loser.

(a) Determine a winning strategy for the first player on a $2 \times n$ grid.

(b) Determine a winning strategy for the first player on an $n \times n$ grid.

(c) Show that there exists a winning strategy for the first player on any size grid apart from 1×1.

3 Amy and Ben play the game of *misère noughts and crosses* on a 3×3 square array. On Amy's turn, she can place an **X** in any vacant square, while on Ben's turn, he can place an **O** in any vacant square. The players take turns to place their symbol, with Amy going first. Any player who gets three in a row (horizontally, vertically or diagonally) immediately loses the game. The game is considered drawn if there is no winner after all squares have been filled.

(a) Which player, if any, has a winning strategy?

(b) Answer the same question if the game is played on a three-dimensional $3 \times 3 \times 3$ cubic array.

4 Two players play a game involving a knight on an 8×8 chessboard. The first player places the knight on the board and the second player makes a knight's move. The two players then take turns to make a knight's move, but may not place the piece on a square it has already visited.

If the player who is unable to move is considered the loser, which player has a winning strategy?

5 Two people play a game involving n coins on a table. The first player takes at least one, but not all, of the coins. The players then take turns to take at least one coin, but no more than was taken on the previous move. The player who takes the last coin is considered the winner.

For which values of n does the second player have a winning strategy?

6 The game of *Domination* is played on an $m \times n$ chessboard. Two players take turns to place a domino so that it covers two adjacent squares of the chessboard. Dominoes are not allowed to overlap and the player who cannot move loses. Dominoes are not allowed to overlap and the player who cannot move loses.

(a) Which player has a winning strategy if Domination is played on a 3×3 board?

(b) Which player has a winning strategy if Domination is played on a $2m \times 2n$ board?

(c) Which player has a winning strategy if Domination is played on a $(2m + 1) \times 2n$ board?

7 At the start of a game, the numbers 1 and 2 are each written 10 times on a blackboard. Two players take turns to erase two of the numbers, replacing them with a 1 if they are different and with a 2 if they are the same. The first player wins if the last number on the board is 1, while the second player wins if it is 2.
Which player has a winning strategy?

8 Initially, the number 2 is written on a blackboard. Two players take turns to erase the number N from the blackboard and replace it with the number $N + d$, where d is one of the divisors of N satisfying $0 < d < N$.
(a) If the player who first writes a number greater than 12345 loses the game, which player has a winning strategy?
(b) If the player who first writes a number greater than 123456 loses the game, which player has a winning strategy?

9 Consider a chocolate bar in the shape of an equilateral triangle, with sides of length n, divided by grid lines into equilateral triangles of side length 1. Two players take turns to break off a triangular piece along one of the grid lines and pass the remaining block of chocolate to the other player. A player who is unable to move or who leaves an equilateral triangle of side length 1 is declared the loser.
For which values of n does the second player have a winning strategy?

10 Ten girls sitting around a circular table are playing a game with N cards. Initially, one girl is holding all of the cards. Each minute, if there is at least one girl holding at least two cards, one of them must pass a card to each of her two neighbours. The game ends if no girl is holding more than one card.
(a) Prove that if $N \geq 10$, then it is impossible for the game to end.
(b) Prove that if $N < 10$, then the game must eventually end.

11 Fifty coins of various denominations lie in a row. Ollie picks up a coin from one end of the row, then Ellie picks up a coin from one end of the row of remaining coins. They alternate in this way until they each have 25 coins.
Prove that Ollie can guarantee to win at least as much money as Ellie.

12 Amy and Ben play a game of 11-in-a-row on an infinite two-dimensional square array. They take turns to choose a vacant square and mark it. Amy goes first and marks squares with an **X**, while Ben marks squares with an **O**. A player wins by being the first to mark 11 consecutive squares vertically, horizontally or diagonally.
(a) Show that Amy can prevent Ben from winning.
(b) Show that Ben can prevent Amy from winning.
(c) Show that (a) and (b) are still true if they are playing 9-in-a-row instead.[1]

[1] For the general case of n-in-a-row, it is known that the first player can force a win for $n \leq 5$ and that the second player can force a draw for $n \geq 8$. As far as we can tell, the status for $n = 6, 7$ is unknown (these cases were open problems as of September 2014).

13 A 16×16 square grid is constructed from sixteen 4×4 smaller square grids called *boxes*. The game of *Ukodus* is played on the 16×16 grid as follows. Two players write, in turn, numbers from the set $\{1, 2, \ldots, 16\}$ in different squares. The numbers in each row, column and box of the 16×16 grid must be different. The loser is the one who is not able to write a number.

Which player has a winning strategy?

14 The game *Hackenbush* takes place on a finite graph with edges coloured either red, blue or black, and a ground node G. There are two players, red and blue. On their turn, a player can choose an edge and remove it from the graph, after which any vertices and edges no longer connected to G. The red player cannot remove blue edges, and the blue player cannot remove red edges.

A player loses if it is their turn and they cannot move.

Assuming red goes first:

(a) For which entirely black graphs does red have a winning strategy?

(b) For which graphs in general does red have a winning strategy?