Competitive Programming and Mathematics Society

Invariants and Semi-Invariants

Workshop 3, Week 8, Term 1, 2021

CPMSoc Mathematics

Table of contents

Invariants

CPMSOC

- An invariant χ is some characteristic of a configuration G that does not change under a particular kind of transformation T, i.e. $\chi(G) = \chi(T(G))$

Invariants

- An invariant χ is some characteristic of a configuration G that does not change under a particular kind of transformation T, i.e. $\chi(G) = \chi(T(G))$
- We say that χ is invariant under T, or with respect to T

Invariants

- An invariant χ is some characteristic of a configuration G that does not change under a particular kind of transformation T, i.e. $\chi(G) = \chi(T(G))$
- We say that χ is invariant under T, or with respect to T
- If two objects have different invariants, neither can be reached by applying T to the other any number of times

Invariants

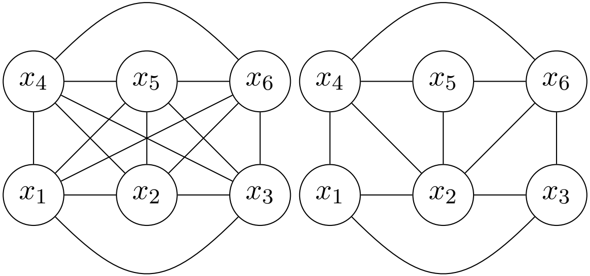
CPMSOC

- An invariant χ is some characteristic of a configuration G that does not change under a particular kind of transformation T, i.e. $\chi(G) = \chi(T(G))$
- We say that χ is invariant under T, or with respect to T
- If two objects have different invariants, neither can be reached by applying T to the other any number of times
- If two objects have the same invariant, there is no guarantee that we can apply T to transform one into the other (nor that we cannot)

Invariants

Example

Show that for any connected planar graph with $V \geq 1$ vertices, E edges and F distinct regions, the equation $V - E + F = 2$ holds.



Invariants

CPMSOC

Example

Show that for any connected planar graph with $V \geq 1$ vertices, E edges and F distinct regions, the equation $V - E + F = 2$ holds.

1. Copy one vertex, v_1, from the given graph, G, into a new plane, P

Invariants

CPMSOC

Example

Show that for any connected planar graph with $V \geq 1$ vertices, E edges and F distinct regions, the equation $V - E + F = 2$ holds.

1. Copy one vertex, v_1, from the given graph, G, into a new plane, P
2. Because the original graph is connected, there was a path from this vertex to any other

Invariants

CPMSOC

Example

Show that for any connected planar graph with $V \geq 1$ vertices, E edges and F distinct regions, the equation $V - E + F = 2$ holds.

1. Copy one vertex, v_1, from the given graph, G, into a new plane, P
2. Because the original graph is connected, there was a path from this vertex to any other
3. For each vertex in G that connected to v_1, create new nodes v_i in P and connect them to v_1 (keeping the new graph planar)

Invariants

CPMSOC

Example

Show that for any connected planar graph with $V \geq 1$ vertices, E edges and F distinct regions, the equation $V - E + F = 2$ holds.

1. Copy one vertex, v_1, from the given graph, G, into a new plane, P
2. Because the original graph is connected, there was a path from this vertex to any other
3. For each vertex in G that connected to v_1, create new nodes v_i in P and connect them to v_1 (keeping the new graph planar)
   - Each new node increases V by 1, decreases E by 1, and leaves F unchanged, so $V - E + F$ is unchanged

Invariants

CPMSOC

Example

Show that for any connected planar graph with $V \geq 1$ vertices, E edges and F distinct regions, the equation $V - E + F = 2$ holds.

1. Copy one vertex, v_1, from the given graph, G, into a new plane, P
2. Because the original graph is connected, there was a path from this vertex to any other
3. For each vertex in G that connected to v_1, create new nodes v_i in P and connect them to v_1 (keeping the new graph planar)
   - Each new node increases V by 1, decreases E by 1, and leaves F unchanged, so $V - E + F$ is unchanged
   - Repeat this for all the nodes just created, and continue until we have all the nodes from the original graph

Invariants

CPMSOC

Example

Show that for any connected planar graph with $V \geq 1$ vertices, E edges and F distinct regions, the equation $V - E + F = 2$ holds.

4. Draw in all the edges not yet copied into P

Invariants

CPMSOC

Example

Show that for any connected planar graph with $V \geq 1$ vertices, E edges and F distinct regions, the equation $V - E + F = 2$ holds.

4. Draw in all the edges not yet copied into P
   - Introducing an edge while keeping the graph planar increases E by 1 and decreases F by 1, so $V - E + F$ is unchanged

Invariants

CPMSOC

Example

Show that for any connected planar graph with $V \geq 1$ vertices, E edges and F distinct regions, the equation $V - E + F = 2$ holds.

4. Draw in all the edges not yet copied into P
   - Introducing an edge while keeping the graph planar increases E by 1 and decreases F by 1, so $V - E + F$ is unchanged
5. We now have a copy of the original graph, and since $V - E + F$ was invariant throughout, it must be the same as it was to begin with, i.e. $1 - 0 + 1 = 2$

Invariants

CPMSOC

Example

Is it possible to tile a 66×62 rectangle with 12×1 rectangles (allowing 90° rotations)?

Invariants

CPMSOC

Example

Is it possible to tile a 66×62 rectangle with 12×1 rectangles (allowing 90° rotations)?

1. Suppose that there is such a tiling. Number the unit squares in the large rectangle as follows.

$$\left[\begin{array}{c|c|c|c|c|c} 1 & 12 & 11 & \cdots & 1 & 12 \\ \hline 2 & 1 & 12 & \cdots & 2 & 1 \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline 5 & 4 & 3 & \cdots & 5 & 4 \\ \hline 6 & 5 & 4 & \cdots & 6 & 5 \end{array}\right]$$

Invariants

CPMSOC

Example

Is it possible to tile a 66×62 rectangle with 12×1 rectangles (allowing 90° rotations)?

1. Suppose that there is such a tiling. Number the unit squares in the large rectangle as follows.

$$\begin{bmatrix} 1 & 12 & 11 & \cdots & 1 & 12 \\ 2 & 1 & 12 & \cdots & 2 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 5 & 4 & 3 & \cdots & 5 & 4 \\ 6 & 5 & 4 & \cdots & 6 & 5 \end{bmatrix}$$

2. Clearly, the area of the rectangle must be divisible by 12.

Invariants

Example

Is it possible to tile a 66×62 rectangle with 12×1 rectangles (allowing 90° rotations)?

1. Suppose that there is such a tiling. Number the unit squares in the large rectangle as follows.

$$\left[\begin{array}{c|c|c|c|c|c} 1 & 12 & 11 & \cdots & 1 & 12 \\ \hline 2 & 1 & 12 & \cdots & 2 & 1 \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline 5 & 4 & 3 & \cdots & 5 & 4 \\ \hline 6 & 5 & 4 & \cdots & 6 & 5 \end{array}\right]$$

2. Clearly, the area of the rectangle must be divisible by 12.
3. Experimenting with smaller $m \times n$ configurations where neither m nor n is a multiple of 12 yet mn is, leads us to conjecture that the 66×62 rectangle cannot be tiled with 12×1 rectangles

Invariants

Example

Is it possible to tile a 66×62 rectangle with 12×1 rectangles (allowing 90° rotations)?

1. Suppose that there is such a tiling. Number the unit squares in the large rectangle as follows.

$$\begin{bmatrix} 1 & 12 & 11 & \cdots & 1 & 12 \\ 2 & 1 & 12 & \cdots & 2 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 5 & 4 & 3 & \cdots & 5 & 4 \\ 6 & 5 & 4 & \cdots & 6 & 5 \end{bmatrix}$$

2. Clearly, the area of the rectangle must be divisible by 12.
3. Experimenting with smaller $m \times n$ configurations where neither m nor n is a multiple of 12 yet mn is, leads us to conjecture that the 66×62 rectangle cannot be tiled with 12×1 rectangles

Invariants

CPMSOC

Example

Is it possible to tile a 66×62 rectangle with 12×1 rectangles (allowing 90° rotations)?

1. Suppose that there is such a tiling. Number the unit squares in the large rectangle as follows.

$$\left[\begin{array}{c|c|c|c|c|c} 1 & 12 & 11 & \cdots & 1 & 12 \\ \hline 2 & 1 & 12 & \cdots & 2 & 1 \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline 5 & 4 & 3 & \cdots & 5 & 4 \\ \hline 6 & 5 & 4 & \cdots & 6 & 5 \end{array}\right]$$

2. Clearly, the area of the rectangle must be divisible by 12.
3. Experimenting with smaller $m \times n$ configurations where neither m nor n is a multiple of 12 yet mn is, leads us to conjecture that the 66×62 rectangle cannot be tiled with 12×1 rectangles

Invariants

CPMSOC

Example

Is it possible to tile a 66×62 rectangle with 121 rectangles (allowing 90° rotations)?

6. Call a colouration of a rectangle "homogenous" if each colour occurs in the same number of squares.

Invariants

CPMSOC

Example

Is it possible to tile a 66×62 rectangle with 121 rectangles (allowing 90° rotations)?

6. Call a colouration of a rectangle "homogenous" if each colour occurs in the same number of squares.
7. We can break up the large rectangle into four sub-rectangles.

$$\left[\begin{array}{c|c} 60 \times 60 & 60 \times 2 \\ \hline 6 \times 60 & 6 \times 2 \end{array}\right]$$

Invariants

CPMSOC

Example

Is it possible to tile a 66×62 rectangle with 121 rectangles (allowing 90° rotations)?

6. Call a colouration of a rectangle "homogenous" if each colour occurs in the same number of squares.
7. We can break up the large rectangle into four sub-rectangles.

$$\left[\begin{array}{c|c} 60 \times 60 & 60 \times 2 \\ \hline 6 \times 60 & 6 \times 2 \end{array}\right]$$

8. It is easy to check that the 60×60, 60×2 and 6×60 sub-rectangles are all homogeneous, since each sub-rectangle has a dimension that is a multiple of 12.

Invariants

CPMSOC

Example

Is it possible to tile a 66×62 rectangle with 121 rectangles (allowing 90° rotations)?

9 But the 6×2 sub-rectangle is coloured as follows:

$$\begin{bmatrix} 1 & 12 \\ \hline 2 & 1 \\ \hline 3 & 2 \\ \hline 4 & 3 \\ \hline 5 & 4 \\ \hline 6 & 5 \end{bmatrix}$$

Invariants

Example

Is it possible to tile a 66×62 rectangle with 121 rectangles (allowing 90° rotations)?

9 But the 6×2 sub-rectangle is coloured as follows:

$$\begin{bmatrix} 1 & 12 \\ \hline 2 & 1 \\ \hline 3 & 2 \\ \hline 4 & 3 \\ \hline 5 & 4 \\ \hline 6 & 5 \end{bmatrix}$$

10 Thus, the larger rectangle is not homogeneous, so the tiling is impossible.

Invariants

CPMSOC

Example

A room begins empty. Each minute, either one person enters or two people leave. After exactly 3^{1999} minutes, could the room contain $3^{1000} + 2$ people?

Invariants

CPMSOC

Example

A room begins empty. Each minute, either one person enters or two people leave. After exactly 3^{1999} minutes, could the room contain $3^{1000} + 2$ people?

1. If there are n people in the room, after one minute, there will be either $n + 1$ or $n - 2$ people. The difference between these two possible outcomes is 3.

Invariants

Example

A room begins empty. Each minute, either one person enters or two people leave. After exactly 3^{1999} minutes, could the room contain $3^{1000} + 2$ people?

1. If there are n people in the room, after one minute, there will be either $n + 1$ or $n - 2$ people. The difference between these two possible outcomes is 3.
2. At any time t, any two possible n-values differ by a multiple of 3.

Invariants

CPMSOC

Example

A room begins empty. Each minute, either one person enters or two people leave. After exactly 3^{1999} minutes, could the room contain $3^{1000}+2$ people?

1. If there are n people in the room, after one minute, there will be either $n+1$ or $n-2$ people. The difference between these two possible outcomes is 3.
2. At any time t, any two possible n-values differ by a multiple of 3.
3. 3^{1999} is a possible n-value after 3^{1999} minutes

Invariants

CPMSOC

Example

A room begins empty. Each minute, either one person enters or two people leave. After exactly 3^{1999} minutes, could the room contain $3^{1000}+2$ people?

1. If there are n people in the room, after one minute, there will be either $n+1$ or $n-2$ people. The difference between these two possible outcomes is 3.
2. At any time t, any two possible n-values differ by a multiple of 3.
3. 3^{1999} is a possible n-value after 3^{1999} minutes
4. $3^{1999} \equiv 0 \pmod 3$ while $3^{1000}+2 \equiv 2 \pmod 3$

Invariants

CPMSOC

Example

A room begins empty. Each minute, either one person enters or two people leave. After exactly 3^{1999} minutes, could the room contain $3^{1000}+2$ people?

1. If there are n people in the room, after one minute, there will be either $n+1$ or $n-2$ people. The difference between these two possible outcomes is 3.
2. At any time t, any two possible n-values differ by a multiple of 3.
3. 3^{1999} is a possible n-value after 3^{1999} minutes
4. $3^{1999} \equiv 0 \pmod 3$ while $3^{1000}+2 \equiv 2 \pmod 3$
5. The room cannot contain $3^{1000}+2$ people after exactly 3^{1999} minutes

Invariants

CPMSOC

Example

A bubble chamber contains three types of subatomic particles: 10 particles of type X, 11 of type Y and 111 of type Z. Whenever an X- and Y-particle collide, they both become Z-particles. Likewise, Y- and Z-particles collide and become X-particles and X- and Z-particles become Y-particles upon collision. Can the particles in the bubble chamber evolve so that only one type is present?

Invariants

CPMSOC

Example

A bubble chamber contains three types of subatomic particles: 10 particles of type X, 11 of type Y and 111 of type Z. Whenever an X- and Y-particle collide, they both become Z-particles. Likewise, Y- and Z-particles collide and become X-particles and X- and Z-particles become Y-particles upon collision. Can the particles in the bubble chamber evolve so that only one type is present?

We will indicate the population at any time by (x, y, z) and transformations by $\rightarrow_{X+Y}, \rightarrow_{Y+Z}, \rightarrow_{X+Z}$.

Invariants

Example

A bubble chamber contains three types of subatomic particles: 10 particles of type X, 11 of type Y and 111 of type Z. Whenever an X- and Y-particle collide, they both become Z-particles. Likewise, Y- and Z-particles collide and become X-particles and X- and Z-particles become Y-particles upon collision. Can the particles in the bubble chamber evolve so that only one type is present?

We will indicate the population at any time by (x, y, z) and transformations by $\rightarrow_{X+Y}, \rightarrow_{Y+Z}, \rightarrow_{X+Z}$.

1. $(10, 11, 111) \rightarrow_{X+Y} (9, 10, 113) \rightarrow_{X+Y} (8, 9, 115) \rightarrow_{X+Z} (7, 11, 114)$.
2. Note that $Z - Y$ goes from 100 to 103, 106 and then 103

Invariants

Example

A bubble chamber contains three types of subatomic particles: 10 particles of type X, 11 of type Y and 111 of type Z. Whenever an X- and Y-particle collide, they both become Z-particles. Likewise, Y- and Z-particles collide and become X-particles and X- and Z-particles become Y-particles upon collision. Can the particles in the bubble chamber evolve so that only one type is present?

We will indicate the population at any time by (x, y, z) and transformations by $\rightarrow_{X+Y}, \rightarrow_{Y+Z}, \rightarrow_{X+Z}$.

1. $(10, 11, 111) \rightarrow_{X+Y} (9, 10, 113) \rightarrow_{X+Y} (8, 9, 115) \rightarrow_{X+Z} (7, 11, 114)$.
2. Note that $Z - Y$ goes from 100 to 103, 106 and then 103
3. $Y - X$ is 1, 1, and 1 again, but then is 4

Invariants

Example

A bubble chamber contains three types of subatomic particles: 10 particles of type X, 11 of type Y and 111 of type Z. Whenever an X- and Y-particle collide, they both become Z-particles. Likewise, Y- and Z-particles collide and become X-particles and X- and Z-particles become Y-particles upon collision. Can the particles in the bubble chamber evolve so that only one type is present?

We will indicate the population at any time by (x, y, z) and transformations by $\rightarrow_{X+Y}, \rightarrow_{Y+Z}, \rightarrow_{X+Z}$.

1. $(10, 11, 111) \rightarrow_{X+Y} (9, 10, 113) \rightarrow_{X+Y} (8, 9, 115) \rightarrow_{X+Z} (7, 11, 114)$.
2. Note that $Z - Y$ goes from 100 to 103, 106 and then 103
3. $Y - X$ is 1, 1, and 1 again, but then is 4
4. Hypothesis: $Z - Y, Y - X, Z - X$ are invariant $\pmod 3$

Invariants

CPMSOC

Example

A bubble chamber contains three types of subatomic particles: 10 particles of type X, 11 of type Y and 111 of type Z. Whenever an X- and Y-particle collide, they both become Z-particles. Likewise, Y- and Z-particles collide and become X-particles and X- and Z-particles become Y-particles upon collision. Can the particles in the bubble chamber evolve so that only one type is present?

We will indicate the population at any time by (x, y, z) and transformations by $\rightarrow_{X+Y}, \rightarrow_{Y+Z}, \rightarrow_{X+Z}$.

6. Note that $(x, y, z) \rightarrow_{X+Y} (x-1, y-1, z+2)$ and so $Z-X$ and $Z-Y$ increase by 3 and $Y-X$ stays the same (similarly for $\rightarrow_{X+Z}$ and $\rightarrow_{Y+Z}$)

Invariants

CPMSOC

Example

A bubble chamber contains three types of subatomic particles: 10 particles of type X, 11 of type Y and 111 of type Z. Whenever an X- and Y-particle collide, they both become Z-particles. Likewise, Y- and Z-particles collide and become X-particles and X- and Z-particles become Y-particles upon collision. Can the particles in the bubble chamber evolve so that only one type is present?

We will indicate the population at any time by (x, y, z) and transformations by $\to_{X+Y}, \to_{Y+Z}, \to_{X+Z}$.

6. Note that $(x, y, z) \to_{X+Y} (x-1, y-1, z+2)$ and so $Z - X$ and $Z - Y$ increase by 3 and $Y - X$ stays the same (similarly for $\to_{X+Z}$ and $\to_{Y+Z}$)
7. For $(10, 11, 111)$, $Y - X = 1 \not\equiv 0 \mod 3$, so the X, Y populations can never be the same

Invariants

CPMSOC

Example

A bubble chamber contains three types of subatomic particles: 10 particles of type X, 11 of type Y and 111 of type Z. Whenever an X- and Y-particle collide, they both become Z-particles. Likewise, Y- and Z-particles collide and become X-particles and X- and Z-particles become Y-particles upon collision. Can the particles in the bubble chamber evolve so that only one type is present?

We will indicate the population at any time by (x, y, z) and transformations by $\to_{X+Y}, \to_{Y+Z}, \to_{X+Z}$.

8. Hence Z cannot contain the whole population; by a similar argument, neither can Y nor X

Invariants

CPMSOC

Example

A bubble chamber contains three types of subatomic particles: 10 particles of type X, 11 of type Y and 111 of type Z. Whenever an X- and Y-particle collide, they both become Z-particles. Likewise, Y- and Z-particles collide and become X-particles and X- and Z-particles become Y-particles upon collision. Can the particles in the bubble chamber evolve so that only one type is present?

We will indicate the population at any time by (x, y, z) and transformations by $\rightarrow_{X+Y}, \rightarrow_{Y+Z}, \rightarrow_{X+Z}$.

8. Hence Z cannot contain the whole population; by a similar argument, neither can Y nor X
9. Therefore, the particles cannot all be of the same type, regardless of the collisions that occur

Invariants

CPMSOC

Example

(IMO 1985)

Consider a set of 1985 positive integers, not necessarily distinct, and none with prime factors bigger than 23. Prove that there must exist four integers in this set whose product is equal to the fourth power of an integer.

Invariants

CPMSOC

Example

(IMO 1985)

Consider a set of 1985 positive integers, not necessarily distinct, and none with prime factors bigger than 23. Prove that there must exist four integers in this set whose product is equal to the fourth power of an integer.

1. Each such number can be written as $k = 2^{f_1} \cdot 3^{f_2} \cdot 5^{f_3} \cdot 7^{f_4} \cdot 11^{f_5} \cdot 13^{f_6} \cdot 17^{f_7} \cdot 19^{f_8} \cdot 23^{f_9}$ where the exponents $f_1, \ldots, f_9$ are nonnegative integers

Invariants

CPMSOC

Example

(IMO 1985)
Consider a set of 1985 positive integers, not necessarily distinct, and none with prime factors bigger than 23. Prove that there must exist four integers in this set whose product is equal to the fourth power of an integer.

1. Each such number can be written as $k = 2^{f_1} \cdot 3^{f_2} \cdot 5^{f_3} \cdot 7^{f_4} \cdot 11^{f_5} \cdot 13^{f_6} \cdot 17^{f_7} \cdot 19^{f_8} \cdot 23^{f_9}$ where the exponents $f_1, \ldots, f_9$ are nonnegative integers
2. If $g_n = \begin{cases} 1 & \text{if } f_n \equiv 1 \pmod 2 \\ 0 & \text{if } f_n \equiv 0 \pmod 2 \end{cases}$, then $k_1 \cdot k_2$ is a perfect square when the g_is match

Invariants

CPMSOC

Arc CLUBS SUPPORTED BY Arc INDEPENDENTLY RUN

Example

(IMO 1985)
Consider a set of 1985 positive integers, not necessarily distinct, and none with prime factors bigger than 23. Prove that there must exist four integers in this set whose product is equal to the fourth power of an integer.

3 There are $2^9 = 512$ possible 9-tuples of the parities of $(f_1, f_2, \ldots f_9)$. By repeated use of the pigeonhole principle, we conclude that 1472 of the integers in the set can be arranged into the 736 pairs

$$(a_1, b_1), (a_2, b_2), \ldots (a_{736}, b_{736})$$

such that each pair contains two numbers with identical 9-tuples of exponent parity.

Invariants

CPMSOC

Example

(IMO 1985)
Consider a set of 1985 positive integers, not necessarily distinct, and none with prime factors bigger than 23. Prove that there must exist four integers in this set whose product is equal to the fourth power of an integer.

3. There are $2^9 = 512$ possible 9-tuples of the parities of $(f_1, f_2, \ldots f_9)$. By repeated use of the pigeonhole principle, we conclude that 1472 of the integers in the set can be arranged into the 736 pairs

$$(a_1, b_1), (a_2, b_2), \ldots (a_{736}, b_{736})$$

such that each pair contains two numbers with identical 9-tuples of exponent parity.

4. Thus the product of the numbers in each pair is a perfect square.

Invariants

CPMSOC

Example

(IMO 1985)
Consider a set of 1985 positive integers, not necessarily distinct, and none with prime factors bigger than 23. Prove that there must exist four integers in this set whose product is equal to the fourth power of an integer.

5. If we let $c_i = a_i b_i$ then each of

$$\sqrt{c_1}, \sqrt{c_2}, \ldots, \sqrt{c_{736}}$$

is an integer with prime factors at most 23.

Invariants

CPMSOC

Example

(IMO 1985)
Consider a set of 1985 positive integers, not necessarily distinct, and none with prime factors bigger than 23. Prove that there must exist four integers in this set whose product is equal to the fourth power of an integer.

5. If we let $c_i = a_i b_i$ then each of

$$\sqrt{c_1}, \sqrt{c_2}, \ldots, \sqrt{c_{736}}$$

is an integer with prime factors at most 23.
6. Using the pigeonhole principle again, we conclude that at least two numbers in the above list share the same 9-tuple of exponent parity.

Invariants

CPMSOC

Example

(IMO 1985)
Consider a set of 1985 positive integers, not necessarily distinct, and none with prime factors bigger than 23. Prove that there must exist four integers in this set whose product is equal to the fourth power of an integer.

5. If we let $c_i = a_i b_i$ then each of

$$\sqrt{c_1}, \sqrt{c_2}, \ldots, \sqrt{c_{736}}$$

is an integer with prime factors at most 23.
6. Using the pigeonhole principle again, we conclude that at least two numbers in the above list share the same 9-tuple of exponent parity.
7. So $\sqrt{c_k}\sqrt{c_j} = n^2$ for some integer n.

Invariants

CPMSOC

Example

(IMO 1985)
Consider a set of 1985 positive integers, not necessarily distinct, and none with prime factors bigger than 23. Prove that there must exist four integers in this set whose product is equal to the fourth power of an integer.

5. If we let $c_i = a_i b_i$ then each of
$$\sqrt{c_1}, \sqrt{c_2}, \ldots, \sqrt{c_{736}}$$
is an integer with prime factors at most 23.
6. Using the pigeonhole principle again, we conclude that at least two numbers in the above list share the same 9-tuple of exponent parity.
7. So $\sqrt{c_k}\sqrt{c_j} = n^2$ for some integer n.
8. Thus $n^4 = c_k c_j = a_j b_j a_k b_k$ so we have found four numbers whose product is a fourth power.

Invariants

Example

Four congruent right triangles are given. One can cut one of them along the altitude and repeat the operation several times with the newly obtained triangles. Prove that no matter how we perform the cuts, we can always find among the triangles two that are congruent.

Invariants

Example

Four congruent right triangles are given. One can cut one of them along the altitude and repeat the operation several times with the newly obtained triangles. Prove that no matter how we perform the cuts, we can always find among the triangles two that are congruent.

1. Without loss of generality, we may assume that the length of the hypotenuse is 1 and the legs are of length p and q.
2. In the process of cutting, the new triangles will be in the ratio $p^m q^n$ to the original ones, for some negative integers n and m. Let's call the pair (m, n).

Invariants

CPMSOC

Example

Four congruent right triangles are given. One can cut one of them along the altitude and repeat the operation several times with the newly obtained triangles. Prove that no matter how we perform the cuts, we can always find among the triangles two that are congruent.

1. Without loss of generality, we may assume that the length of the hypotenuse is 1 and the legs are of length p and q.
2. In the process of cutting, the new triangles will be in the ratio $p^m q^n$ to the original ones, for some negative integers n and m. Let's call the pair (m, n).
3. Each time we cut a triangle, we replace the pair (m, n) with the pairs $(m + 1, n)$ and $(m, n + 1)$.

Invariants

Example

Four congruent right triangles are given. One can cut one of them along the altitude and repeat the operation several times with the newly obtained triangles. Prove that no matter how we perform the cuts, we can always find among the triangles two that are congruent.

1. Without loss of generality, we may assume that the length of the hypotenuse is 1 and the legs are of length p and q.
2. In the process of cutting, the new triangles will be in the ratio $p^m q^n$ to the original ones, for some negative integers n and m. Let's call the pair (m, n).
3. Each time we cut a triangle, we replace the pair (m, n) with the pairs $(m+1, n)$ and $(m, n+1)$.
4. Assign the weight $\frac{1}{2^{m+n}}$ to the pair (m, n). Then the sum I of all weights is invariant under cuts.

Invariants

CPMSOC

Example

Four congruent right triangles are given. One can cut one of them along the altitude and repeat the operation several times with the newly obtained triangles. Prove that no matter how we perform the cuts, we can always find among the triangles two that are congruent.

5 The initial value of I is 4. If at some stage the triangles were pairwise incongruent, then the value of I would be strictly less than

$$\sum_{m,n=0}^{\infty} \frac{1}{2^{m+n}} = \sum_{m=0}^{\infty} \frac{1}{2^m} \sum_{n=0}^{\infty} \frac{1}{2^n} = 4,$$

which is a contradiction. Hence any resulting configuration must contain two congruent triangles.

Invariants

CPMSOC

Example

There are n markers, each with one side white and the other side black, aligned in a row with their white sides up. At each step, if possible, we choose a marker with the white side up (but not one of the outermost markers), remove it, and reverse the two neighbouring markers. Prove that one can reach a configuration with only two markers left if and only if $n-1$ is not divisible by 3.

Invariants

Example

There are n markers, each with one side white and the other side black, aligned in a row with their white sides up. At each step, if possible, we choose a marker with the white side up (but not one of the outermost markers), remove it, and reverse the two neighbouring markers. Prove that one can reach a configuration with only two markers left if and only if $n - 1$ is not divisible by 3.

1. We refer to a marker by the colour of its visible face.

Invariants

CPMSOC

Example

There are n markers, each with one side white and the other side black, aligned in a row with their white sides up. At each step, if possible, we choose a marker with the white side up (but not one of the outermost markers), remove it, and reverse the two neighbouring markers. Prove that one can reach a configuration with only two markers left if and only if $n-1$ is not divisible by 3.

1. We refer to a marker by the colour of its visible face.
2. Note that the parity of the number of black markers must remain unchanged during the game.

Invariants

CPMSOC

Example

There are n markers, each with one side white and the other side black, aligned in a row with their white sides up. At each step, if possible, we choose a marker with the white side up (but not one of the outermost markers), remove it, and reverse the two neighbouring markers. Prove that one can reach a configuration with only two markers left if and only if $n - 1$ is not divisible by 3.

1. We refer to a marker by the colour of its visible face.
2. Note that the parity of the number of black markers must remain unchanged during the game.
3. Hence, if only two markers are left then they must be the same colour.

Invariants

CPMSOC

Example

There are n markers, each with one side white and the other side black, aligned in a row with their white sides up. At each step, if possible, we choose a marker with the white side up (but not one of the outermost markers), remove it, and reverse the two neighbouring markers. Prove that one can reach a configuration with only two markers left if and only if $n-1$ is not divisible by 3.

1. We refer to a marker by the colour of its visible face.
2. Note that the parity of the number of black markers must remain unchanged during the game.
3. Hence, if only two markers are left then they must be the same colour.
4. We define an invariant as follows. To a white marker with t black markers to its left we assign the number $(-1)^t$.

Invariants

CPMSOC

Example

There are n markers, each with one side white and the other side black, aligned in a row with their white sides up. At each step, if possible, we choose a marker with the white side up (but not one of the outermost markers), remove it, and reverse the two neighbouring markers. Prove that one can reach a configuration with only two markers left if and only if $n-1$ is not divisible by 3.

Invariants

CPMSOC

Example

There are n markers, each with one side white and the other side black, aligned in a row with their white sides up. At each step, if possible, we choose a marker with the white side up (but not one of the outermost markers), remove it, and reverse the two neighbouring markers. Prove that one can reach a configuration with only two markers left if and only if $n-1$ is not divisible by 3.

5. The invariant S is the residue class modulo 3 of the sum of all the numbers assigned to the white markers.

Invariants

CPMSOC

Example

There are n markers, each with one side white and the other side black, aligned in a row with their white sides up. At each step, if possible, we choose a marker with the white side up (but not one of the outermost markers), remove it, and reverse the two neighbouring markers. Prove that one can reach a configuration with only two markers left if and only if $n-1$ is not divisible by 3.

5. The invariant S is the residue class modulo 3 of the sum of all the numbers assigned to the white markers.
6. We can check that S is invariant under the operation defined in the statement. For instance, if a white marker with t black markers on the left and whose neighbours are both black is removed, then S increases by $-(-1)^t + (-1)^{t-1} + (-1)^{t-1} = 3(-1)^{t-1}$, which is zero modulo 3. The other three cases are analogous.

Invariants

CPMSOC

Example

There are n markers, each with one side white and the other side black, aligned in a row with their white sides up. At each step, if possible, we choose a marker with the white side up (but not one of the outermost markers), remove it, and reverse the two neighbouring markers. Prove that one can reach a configuration with only two markers left if and only if $n-1$ is not divisible by 3.

Invariants

CPMSOC

Example

There are n markers, each with one side white and the other side black, aligned in a row with their white sides up. At each step, if possible, we choose a marker with the white side up (but not one of the outermost markers), remove it, and reverse the two neighbouring markers. Prove that one can reach a configuration with only two markers left if and only if $n-1$ is not divisible by 3.

7. If the game ends with two black markers then S is zero,; if it ends with two white markers then S is 2. This proves that $n-1$ is not divisible by 3, as S is initially congruent to n modulo 3.

Invariants

CPMSOC

Example

There are n markers, each with one side white and the other side black, aligned in a row with their white sides up. At each step, if possible, we choose a marker with the white side up (but not one of the outermost markers), remove it, and reverse the two neighbouring markers. Prove that one can reach a configuration with only two markers left if and only if $n-1$ is not divisible by 3.

7. If the game ends with two black markers then S is zero,; if it ends with two white markers then S is 2. This proves that $n-1$ is not divisible by 3, as S is initially congruent to n modulo 3.
8. Conversely, if we start with $n \geq 5$ white markers, then n is congruent to 0 or 2 modulo 3. Then by removing in three consecutive moves the leftmost allowed white markers, we obtain a row of $n-3$ white markers. Working backwards, we can reach either 2 or 3 white markers. In the latter case, one more move gives 2 black markers as desired.

Semi-invariants

- A semi-invariant (or monovariant) ψ is some characteristic of a configuration G that only increases (or decreases) under a particular kind of transformation T, i.e. $\psi(G) > \psi(T(G))$ or $\psi(G) \geq \psi(T(G))$

Semi-invariants

CPMSOC

- A semi-invariant (or monovariant) ψ is some characteristic of a configuration G that only increases (or decreases) under a particular kind of transformation T, i.e. $\psi(G) > \psi(T(G))$ or $\psi(G) \geq \psi(T(G))$
- It may be stricly or non-strictly increasing

Semi-invariants

- A semi-invariant (or monovariant) ψ is some characteristic of a configuration G that only increases (or decreases) under a particular kind of transformation T, i.e. $\psi(G) > \psi(T(G))$ or $\psi(G) \geq \psi(T(G))$
- It may be stricly or non-strictly increasing
- If $\psi(A) \geq \psi(B)$, then B cannot be written in the form $T(T(\ldots T(A)))$