Competitive Programming and Mathematics Society

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Elementary Algebra

CPMSoc Mathematics Team

Welcome

- We will announce the winner to the Welcome Week Competition!
- All workshops shall be 2 hours long.
- The notes of the contents in the workshops shall be provided on the CPMSoc website.
- Each workshop will have an accompanying problem set, which can be found in the notes.
- There will be workshops on odd numbered weeks from week 5 on wards (That's this week).
- Hope you enjoy yourselves and feel free to ask questions during the workshops ☺.

Attendance

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Table of contents

Winners

Winners

1. Cyril Subramanian (Congratulations ☺)
2. Ryno
3. ChestaWu

Random Draws

The recipients of the 2 Random Prizes will be selected after the other prizes are awarded, and are to be determined by performing the following procedure 2 times:

Define a "selectable" participant as one who is eligible and who has not already been awarded a prize. Let the total number of selectable participants be T. If $T = 0$, no prize is awarded. Otherwise, one selectable participant is randomly selected to win a Merit Prize, such that for each selectable participant, they have a $\frac{1}{T}$ chance of being selected.

Note: All prize recipients will be contacted via their preferred email very soon with details of how to claim their prizes.

Debrief

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Example (Question 20)

Let $f : \mathbb{R} \to \mathbb{R}$ be such that

$$f\left(2x - \frac{f(x)}{\alpha}\right) = \alpha x, \forall\ x \in \mathbb{R}.$$

Prove that $f(x) = \alpha(x - c)$, for all $x \in \mathbb{R}$.
Note: $\alpha \neq 0$.

Notation

1. $\mathbb{N} = \{1, 2, 3 \cdots\}$
2. $\mathbb{P}_n = \{p \in \mathbb{P} : p|n, \ n \in \mathbb{N}\}$
3. Any problem in the problem section that is starred (*) is a standard theorem as well and therefore is highly recommended to be learnt.

Algebraic Identities

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Familiarity with Algebraic Identities is one of the basic skills that is required when one indulges in Mathematics nevertheless Competitive Mathematics. In this section we present a treatment of some of the most important algebraic identities that one must know.

Lemma (Sophie Germain)

Let $a, b \in \mathbb{R}$ *then*

$$a^4 + 4b^4 = (a^2 - 2ab + 2b^2)(a^2 + 2ab + 2b^2).$$

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Proof.

$$a^4 + 4b^4 + 4a^2b^2 - 4a^2b^2 = (a^2 + 2b^2)^2 - 4a^2b^2 = (a^2 - 2ab + 2b^2)(a^2 + 2ab + 2b^2).$$

∎

Application

Example

Given two line segments of lengths a and b, construct with a straightedge and compass a segment of length $\sqrt[4]{a^4 + b^4}$.

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Given two line segments of length a and b, we have

$$a^4 + b^4 = (a^2 - \sqrt{2}ab + b^2)(a^2 + \sqrt{2}ab + 2b^2),$$

hence

$$\sqrt[4]{a^4 + b^4} = \sqrt[4]{(a^2 - \sqrt{2}ab + b^2)(a^2 + \sqrt{2}ab + 2b^2)}.$$

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$$\sqrt[4]{a^4 + b^4} = \sqrt[4]{(a^2 - \sqrt{2}ab + b^2)(a^2 + \sqrt{2}ab + 2b^2)}.$$

Through the law of cosines, we can construct segments of length $\sqrt{a^2 \pm \sqrt{2}ab + b^2}$ using triangles of side a and b with the angle between them being 135 and 45 respectively. Subsequently we can also construct $\sqrt{xy}$ for "constructible" x and y as this is nothing but the geometric mean given by AD in a right angled triangle ABC (angle(A)=90) with $BD = x$ and $CD = y$. ∎

Algebraic Identities

Another important Identity that one might be familiar with is

Lemma

Let $a, b, c \in \mathbb{R}$, *then* $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$.

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Proof.

Consider the following

$$D = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix},$$

which is to evaluate in two ways first we take the determinant the usual way using Sarrus' rule, and then by adding all the rows and factoring $(a + b + c)$. ■

Application

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Consider through AM-GM,

$$a^2 + b^2 \geq 2ab,$$
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$$c^2 + a^2 \geq 2ca.$$

Application

Proof.

Consider $a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$. We take note that $a+b+c \geq 0$ for all $a,b,c \geq 0$. Hence all that remains to be proven is that $(a^2+b^2+c^2-ab-bc-ca) \geq 0$.

Consider through AM-GM,

$$a^2+b^2 \geq 2ab,$$
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Adding all of the above proves the point. A more direct way is to notice that $(a^2+b^2+c^2-ab-bc-ca)=\frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2]$ which is manifestly non-negative. ∎

Fundamental Theorem of Algebra

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Lemma

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A corollary of this theorem is that every polynomial with complex coefficients can be written as the product of linear factors with complex in the form $P(z) = A(x - z_1)(x - z_2)\ldots(x - z_n)$ where $z_1, z_2, \ldots, z_n$ are the complex roots (there can be repeated roots, this is referred to as the **multiplicity** of the roots).

Polynomial Division

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Lemma (Remainder Theorem)

If $A(x)$ and $B(x)$ are polynomials with real coefficients then there exist polynomials $Q(x)$ and $R(x)$ such that

$$A(x) = B(x)Q(x) + R(x)$$

where $\deg(R) < \deg(B)$.

*Note: Q and R are respectively called (**quotient**) and (**remainder**).*

The above lemma about polynomial division is analogous to the ideas of quotient and remainders we are familiar with when we divide two integers.

Factor Theorem

Now we move onto the factor theorem, which is a simple way of searching for linear factors of polynomials.

Lemma

If $p \in \mathbb{R}[x]$ *then* $p(a) = 0$ *for some* $a \in \mathbb{R}$ *if and only if* $x - a$ *is a factor of* $p(x)$.

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Proof.

If we divide the polynomial $p(x)$ by $x - a$ then we can write $p(x)$ in the form $p(x) = (x - a)q(x) + c$ where $q(x)$ is a polynomial and c is a constant. Note that $p(a) = c$ and we see that $c = 0$ is equivalent to both $p(a) = 0$ and $x - a$ being a factor of $p(x)$. ■

Applications

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Example

Let $A, B, C, D \in \mathbb{R}[x]$ such that

$$A(x^5) + xB(x^5) + x^2C(x^5) = (1 + x + x^2 + x^3 + x^4)D(x), \forall x \in \mathbb{R}.$$

Prove that $(x - 1)$ is a factor of A.

Application

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Proof.

Consider $x = \omega, \omega^2, \omega^3$, where ω is the fifth root of unity we get

$$A(1) + \omega B(1) + \omega^2 C(1) = 0$$
$$A(1) + \omega^2 B(1) + \omega^4 C(1) = 0$$
$$A(1) + \omega^3 B(1) + \omega^6 C(1) = 0$$

therefore we have that $A(1) = B(1) = C(1) = 0$ by solving the simultaneous equations and therefore using the factor theorem we have that $x - 1$ is a factor of $A(x)$. ∎

Triangle Inequality

The triangle inequality is a simple but very useful inequality involving absolute values.

Lemma (Triangle Inequality)

Let $x, y \in \mathbb{R}$. *Then*

$$|x + y| \leq |x| + |y|, \ \forall x, y \in \mathbb{R}.$$

Triangle Inequality

The triangle inequality is a simple but very useful inequality involving absolute values.

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Let $x, y \in \mathbb{R}$. *Then*

$$|x + y| \leq |x| + |y|, \;\; \forall x, y \in \mathbb{R}.$$

Proof.

Squaring both sides means that the inequality is equivalent to

$$x^2 + y^2 + 2xy \leq x^2 + y^2 + 2|xy|,$$

which is true because $|xy| \geq xy$. ∎

Corollaries

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Corollary

We can prove a generalization too using induction, that

$$|x_1 + x_2 + \cdots + x_n| \le |x_1| + |x_2| + \cdots + |x_n|, \ \forall x_i \in \mathbb{R}.$$

Corollaries

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Corollary (Complete Triangle Inequality)

Let $x, y \in \mathbb{R}$. *Then*

$$||x| - |y|| \leq |x + y| \leq |x| + |y|, \ \forall x, y \in \mathbb{R}.$$

AM-GM Inequalities

One of the other inequalities that is quite often usefull in mathematics is the Arithmetic-Geometric Inequality or commonly known as AM-GM.

Lemma (AM-GM Inequality)

Let $x_1, \cdots, x_n$ *be positive reals. Then*

$$\frac{x_1 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdots x_n}.$$

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$$\frac{x_1 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdots x_n}.$$

Proof.

We begin by using the fact that

$$\log(x) \leq x - 1, \forall x > 0.$$

Which follows due to the MVT (Try this). Let $a_1, \cdots, a_n$ be positive reals and define A to be the arithematic mean i.e.

$$A = \frac{a_1 + \cdots + a_n}{n}.$$

Consider $x = \frac{a_i}{A}$, therefore

$$\log\left(\frac{a_i}{A}\right) \leq \frac{a_i}{A} - 1 \implies \log\left(\frac{a_1 a_2 \ldots a_n}{A^n}\right) \leq 0.$$

■

Weighted AM-GM

There are more general versions of many classical inequalities that involves weights.

Lemma (Weighted AM-GM Inequality)

For positive real numbers $a_1, a_2, \ldots, a_n$ and positive real numbers $w_1, w_2, \ldots, w_n$ (called weights), we have the inequality

$$\frac{w_1 a_1 + w_2 a_2 + \ldots + w_n a_n}{w_1 + w_2 + \ldots + w_n} \geq \sqrt[w_1+w_2+\ldots+w_n]{a_1^{w_1} a_2^{w_2} \ldots a_n^{w_n}}$$

with equality if and only if $a_1 = a_2 = \ldots = a_n$.

Rearrangement Inequality

A useful inequality that involves sequences of numbers is the **rearrangement inequality**, which aims to maximise or minimise the sum of products of corresponding terms in two sequences.

Lemma (Rearrangement inequality)

Let $(a_1, a_2, \cdots, a_n)$ *and* $(x_1, x_2, \cdots, x_n)$ *be two sequences of real numbers. Then the permutation* $(b_1, b_2, \cdots, b_n)$ *of* $(x_1, x_2, \cdots, x_n)$ *which maximises the expression*

$$E = a_1b_1 + a_2b_2 + \ldots + a_nb_n$$

is the permutation where the sequences $(a_1, a_2, \ldots, a_n)$ *and* $(b_1, b_2, \ldots, b_n)$ *are sorted the same way. The permutation that minimises the expressions is where the sequences are sorted the opposite way.*

Cauchy-Schwarz Inequality

The **Cauchy-Schwarz** inequality is another powerful inequality that involves two sequences of real numbers and has many generalisations.

Lemma (Cauchy-Schwarz inequality)

If $(x_1, x_2, \ldots, x_n)$ *and* $(y_1, y_2, \ldots, y_n)$ *are two sequences of real numbers, then*

$$(x_1^2 + x_2^2 + \ldots + x_n^2)(y_1^2 + y_2^2 + \ldots + y_n^2) \geq (x_1 y_1 + x_2 y_2 + \ldots + x_n y_n)^2$$

Equality holds if and only if we have the equal ratios

$$\frac{x_1}{y_1} = \frac{x_2}{y_2} = \cdots = \frac{x_n}{y_n}$$

Example Problems

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1. (2004 Russian Mathematics Olympiad) (Beginner) Let a, b, c be positive numbers, satisfying $a + b + c = \frac{\pi}{2}$, prove that

$$\cos(a) + \cos(b) + \cos(c) \geq \sin(a) + \sin(b) + \sin(c).$$

2. If real numbers x and y satisfy the condition $x^2 + xy + y^2 = 1$, find the minimum and maximum value of $x^3y + xy^3$.
3. Let a, b, c, d be real numbers such that $b - d \geq 5$ and all zeros x_1, x_2, x_3, and x_4 of the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ are real. Then find the minimum of

$$(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1).$$

4. Let $a_1, a_2, \ldots, a_n$ $(n > 3)$ be real numbers such that

$$a_1 + a_2 + \cdots + a_n \geq n \qquad \text{and} \qquad a_1^2 + a_2^2 + \cdots + a_n^2 \geq n^2.$$

Prove that $\max(a_1, a_2, \ldots, a_n) \geq 2$.